—*Brief Paper*—

RECEDING HORIZON H_∞ CONTROL FOR SYSTEMS WITH A STATE-DELAY

Young Sam Lee, Soo Hee Han, and Wook Hyun Kwon

ABSTRACT

This paper proposes the receding horizon H_∞ control (RHHC) for linear systems with a state-delay. We first proposes a new cost function for a finite horizon dynamic game problem. The proposed cost function includes two terminal weighting terms, each of which is parameterized by a positive definite matrix, called a terminal weighting matrix. Secondly, we derive the RHHC from the solution to the finite dynamic game problem. Thirdly, we propose an LMI condition under which the saddle point value satisfies the nonincreasing monotonicity. Finally, we show the asymptotic stability and H_∞-norm boundedness of the closed-loop system controlled by the proposed RHHC. Through a numerical example, we show that the proposed RHHC is stabilizing and satisfies the infinite horizon H_∞-norm bound.

KeyWords: Receding horizon H_∞ control (RHHC), terminal weighting matrix, nonincreasing monotonicity, H_∞-norm bound, saddle point value.

I. INTRODUCTION

Receding horizon controls (RHC) have attracted much attention from academia and industry because of its ability to handle input constraint, disturbance, time-varying tracking commands, and nonlinear systems [1-3]. Most research results established on the RHC up to now are concentratedon ordinary systems without delays. Receding horizon approaches for systems with a state-delay can be found only recently [4,5]. A simple control method based on the receding horizon concept is proposed for state-delayed systems in [4]. However, it does not have a state weighting in the cost function unlike the normal RHC. Furthermore, the stability can be checked only after the RHC solution is obtained. General extension of the RHC to state-delayed systems appears in [5]. RHC proposed there has state weighting in the cost function. Furthermore, it has the guaranteed closed-loop stability if the optimal cost satisfies nonincreasing monotonicity.

The RHC for ordinary systems has been extended to H_∞ problems in order to combine the practical advantage of the RHC with the robustness of the H_∞ control [6-8]. The saddle point value in H_∞ problems corresponds to the optimal cost in LQ problems. Those results in [6-8] propose conditions for nonincreasing monotonicity of the saddle point value. Extension to linear time-varying (LTV) systems were presented in [9,10]. For state-delayed systems, there have been many approaches for H_∞ problems [11-13]. Among them, the results in [12] consider a finite horizon H_∞ control problems. However, since it deals with finite horizon problem, the closed-loop stability issue was not considered.

To the best of our knowledge, there exists no theory developed for receding horizon H_∞ control (RHHC) for state-delayed systems. The purpose of this paper is to lay the cornerstone of the theory on RHHC for state-delayed systems. The issues such as solution, stability, existence condition, norm boundedness will be addressed in the main results. However, we do not argue that the proposed control method has any advantage over exiting H_∞ control method

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in terms of performance index.

The rest of this paper is structured as follows: In Section 2, we obtain a solution to a receding horizon H_∞ control problem. In Section 3, we derive an LMI condition, under which nonincreasing monotonicity condition of a saddle point value holds. In Section 4, we show that the proposed RHHC has asymptotic stability and satisfies H_∞-norm boundedness. In Section 5, we provide a numerical example to illustrate that the proposed RHHC is stabilizing as well as guarantees the H_∞-norm bound. Finally, we make conclusions in Section 6.

Throughout the paper, the notation $P > 0$ ($P \geq 0$) implies that the matrix P is symmetric and positive definite (positive semi-definite). Similarly, $P < 0$ ($P \leq 0$) implies that the matrix P is symmetric and negative definite (negative semi-definite). '★' is used to denote elements under the main diagonal of a symmetric matrix. $L_2[0, \infty)$ and $L_2[t_0, t_f]$ denotes the space of square integrable functions on $[0, \infty)$ and $[t_0, t_f]$, respectively.

II. RECEDING HORIZON H_∞ CONTROL

Consider a linear time-invariant system with a state-delay

$$\dot{x}(t) = A\,x(t) + A_1\,x(t-h) + B\,u(t) + B_w\,w(t) \tag{1}$$

$$z(t) = C\,x(t) + D\,u(t) \tag{2}$$

where $x \in \mathbf{R}^n$ is the state, $u \in \mathbf{R}^m$ is the control input, $w \in \mathbf{R}^l$ is the disturbance signal which belongs to $L_2[0, \infty)$, $z \in \mathbf{R}^p$ is the controlled output, $h > 0$ is the constant delay, respectively. It is assumed that $C^TD = 0$ and $D^TD = I$. In order to obtain RHHC, we will first consider the finite-horizon cost function as follows:

$$\begin{aligned} &J(x_{t_0}, t_0, t_f, u, w) \\ &= \int_{t_0}^{t_f} [x^T(\tau)\,Q\,x(\tau) + u^T(\tau)\,u(\tau) - \gamma^2 w^T(\tau)\,w(\tau)]\,d\tau \\ &\quad + x^T(t_f)\,F_1\,x(t_f) + \int_{t_f-h}^{t_f} x^T(\tau)\,F_2\,x(\tau)\,d\tau, \end{aligned}$$

where $Q = C^TC > 0$, $F_1 > 0$, $F_2 > 0$. We can regard J as a function of either L_2 signals or feedback strategies. Let $\mathcal{M} = \{\mu: [t_0, t_f] \times \mathbf{C}_n[-h, 0] \to \mathbf{R}^m\}$ and $\mathcal{N} = \{\nu: [t_0, t_f] \times \mathbf{C}_n[-h, 0] \to \mathbf{R}^l\}$, where $\mathbf{C}_n[-h, 0]$ is the space of n-dimensional vector functions continuous on $[-h, 0]$. Spaces $\mathcal{M}$ and $\mathcal{N}$ are strategy spaces, and we will write strategies as μ, ν to distinguish them from signals u, w. Denote $x_t = x(t + \theta)$, $h \in [-h, 0]$. Therefore $x_t \in \mathbf{C}_n[-h, 0]$ by the definition of $\mathbf{C}_n[-h, 0]$.

Let's formulate a dynamic game problem

$$\min_{\mu \in \mathcal{M}} \max_{\nu \in \mathcal{N}} J(x_{t_0}, t_0, t_f, \mu, \nu), \tag{3}$$

which is a zero sum game, where u is the minimizing player and w is the maximizing player. The optimal u and the worst case w are called saddle point strategies. A saddle point solution $u(\tau) = u^*(\tau, x(\tau))$, $w(\tau) = \nu^*(\tau, x(\tau))$ satisfies

$$\begin{aligned} &J(x_{t_0}, t_0, t_f, \mu^*, w) \\ &\leq J(x_{t_0}, t_0, t_f, \mu^*, \nu^*) \\ &\leq J(x_{t_0}, t_0, t_f, u, \nu^*), \quad \forall u, w \in L_2[t_0, t_f] \end{aligned}$$

The value $J(x_{t_0}, t_0, t_f, \mu^*, \nu^*)$ is called the saddle point value. For simple notation, the saddle point value will be denoted by $J^*(x_{t_0}, t_0, t_f)$ throughout this paper, *i.e.*,

$$J^*(x_{t_0}, t_0, t_f) \triangleq J(x_{t_0}, t_0, t_f, \mu^*, \nu^*).$$

The purpose of this paper is to develop a method to design a control law, u_R, based on the receding horizon concept such that

(a) in case of zero disturbance, the closed-loop system is asymptotically stable

(b) with zero initial condition, the closed-loop transfer function from w to z, *i.e.*, T_{zw}, satisfies the H_∞-norm bound, for given $\gamma > 0$,

$$\| T_{zw} \|_\infty \leq \gamma. \tag{4}$$

Since the control is based on the receding horizon strategy and the closed-loop system satisfies the H_∞-norm bound, we will call it receding horizon H_∞ control (RHHC).

Remark 2.1. It is noted that the terminal weighting function consists of two terms, parameterized by two matrices F_1 and F_2. We will call them terminal weighting matrices in this paper. The purpose of adding a second terminal weighting term, parameterized by F_2, is to take the delay effect into account such that the resulting RHHC is stabilizing. More specifically, if F_2 is chosen properly, the saddle point value satisfy the well-known 'nonincreasing monotonicity property', which will be considered in Section 3.

Before moving on, we introduce a lemma, which establishes a sufficient condition for a control u and a disturbance w to be saddle point strategies. In the lemma, $V(\tau, x_\tau) : [t_0, t_f] \times \mathbf{C}_n[-h, 0] \to \mathbf{R}$ denotes a continuous and differentiable functional. Furthermore, we will use the notation

$$\frac{d}{d\tau} V(\tau, x_\tau)\big|_{\substack{\mu(\tau, x_\tau) \\ \nu(\tau, x_\tau)}} \triangleq \lim_{\Delta\tau \to 0} \left[\frac{V(\tau + \Delta\tau, x_{\tau+\Delta\tau}^{\mu,\nu}) - V(\tau, x_\tau)}{\Delta\tau} \right]$$

where $x_{\tau+\Delta\tau}^{\mu,\nu} = x(\tau + \Delta\tau + s),\ s \in [-h, 0]$ is the solution of the system (1) resulting from the control $u(t) = \mu(t, x_\tau)$ and disturbance $w(t) = \nu(t, x_\tau)$.

Lemma 2.1. Assume that there exists a continuous functional $V(\tau, x_\tau) : [t_0, t_f] \times \mathbf{C}_n[-h, 0] \to \mathbf{R}$, and a vector functional $\mu^*(\tau, x_\tau) : [t_0, t_f] \times \mathbf{C}_n[-h, 0] \to \mathbf{R}^m$ and $\nu^*(\tau, x_\tau) : [t_0, t_f]$

$\times \mathbf{C}_n[-h, 0] \to \mathbf{R}^l$ such that

(a) $V(t_f, x_{t_f})$

$$= x^T(t_f)\, F_1\, x(t_f) + \int_{t_f-h}^{t_f} x^T(\tau)\, F_2\, x(\tau)\, d\tau$$

(b) $$\frac{d}{d\tau} V(\tau, x_\tau)\Big|_{\substack{\mu^*(\tau,x_\tau)\\ \nu^*(\tau,x_\tau)}} + x^T(\tau)\, Q\, x(\tau)$$

$$+ \mu^{*T}(\tau, x_\tau)\, \mu^*(\tau, x_\tau) - \gamma^2 \nu^{*T}(\tau, x_\tau)\, \nu^*(\tau, x_\tau) = 0$$

(c) $$\frac{d}{d\tau} V(\tau, x_\tau)\Big|_{\substack{\mu^*(\tau,x_\tau)\\ \nu(\tau,x_\tau)}} + x^T(\tau)\, Q\, x(\tau)$$

$$+ \mu^{*T}(\tau, x_\tau)\, \mu^*(\tau, x_\tau) - \gamma^2 \nu^T(\tau, x_\tau)\, \nu(\tau, x_\tau)$$

$$\le \frac{d}{d\tau} V(\tau, x_\tau)\Big|_{\substack{\mu^*(\tau,x_\tau)\\ \nu^*(\tau,x_\tau)}} + x^T(\tau)\, Q\, x(\tau)$$

$$+ \mu^{*T}(\tau, x_\tau)\, \mu^*(\tau, x_\tau) - \gamma^2\, \nu^{*T}(\tau, x_\tau)\, \nu^*(\tau, x_\tau)$$

$$\le \frac{d}{d\tau} V(\tau, x_\tau)\Big|_{\substack{\mu(\tau,x_\tau)\\ \nu^*(\tau,x_\tau)}} + x^T(\tau)\, Q\, x(\tau)$$

$$+ \mu^T(\tau, x_\tau)\, \mu(\tau, x_\tau) - \gamma^2\, \nu^{*T}(\tau, x_\tau)\, \nu^*(\tau, x_\tau)$$

for all $\tau \in [t_0, t_f]$ and all $x_t \in \mathbf{C}_n[-h, 0]$. Then, $V(s, x_s) = J(x_s, s, t_f, \mu^*, \nu^*)$ and

$$J(x_s, s, t_f, \mu^*, \nu) \le J(x_s, s, t_f, \mu^*, \nu^*) \le J(x_s, s, t_f, \mu, \nu^*) \tag{5}$$

for all $s \in [t_0, t_f]$. That is, $u(\tau) = \mu^*(\tau, x_\tau)$ and $w(\tau) = \nu^*(\tau, x_\tau)$ are saddle point solutions and $V(\tau, x_\tau)$ is a saddle point value.

Proof. Integrating (b) from s to t_f yields

$$V(t_f, x_{t_f}) - V(s, x_s) + \int_s^{t_f} [x^T(\tau)\, Q\, x(\tau) + \mu^{*T}(\tau, x_\tau)\, \mu^*(\tau, x_\tau) - \gamma^2 \nu^{*T}(\tau, x_\tau)\, \nu^*(\tau, x_\tau)]\, d\tau = 0$$

Substituting $V(t_f, x_{t_f})$ in (a) into the above equation yields

$$V(s, x_s) = \int_s^{t_f} [x^T(\tau)\, Q\, x(\tau) + \mu^{*T}(\tau, x_\tau)\, \mu^*(\tau, x_\tau) - \gamma^2\, \nu^{*T}(\tau, x_\tau)\, \nu^*(\tau, x_\tau)]\, d\tau + x^T(t_f)\, F_1\, x(t_f) + \int_{t_f-h}^{t_f} x^T(\tau)\, F_2\, x(\tau)\, d\tau = J(x_s, s, t_f, \mu^*, \nu^*)\,.$$

From the inequality relation in (c), we obtain

$$\frac{d}{d\tau} V(\tau, x_\tau)\Big|_{\substack{\mu(\tau,x_\tau)\\ \nu^*(\tau,x_\tau)}} + x^T(\tau)\, Q\, x(\tau) + \mu^T(\tau, x_\tau)\, \mu(\tau, x_\tau) - \gamma^2\, \nu^{*T}(\tau, x_\tau)\, \nu^*(\tau, x_\tau) \ge 0\,.$$

Integrate the above inequality from s to t_f. After some manipulation, we lead to

$$V(s, x_s) \le \int_{t_0}^{t_f} [x^T(\tau)\, Q\, x(\tau) + \mu^T(\tau, x_\tau)\, \mu(\tau, x_\tau) - \gamma^2\, \nu^{*T}(\tau, x_\tau)\, \nu^*(\tau, x_\tau)]\, d\tau + x^T(t_f)\, F_1\, x(t_f) + \int_{t_f-h}^{t_f} x^T(\tau)\, F_2\, x(\tau)\, d\tau = J(x_s, s, t_f, \mu, \nu^*)\,.$$

Similarly, we have

$$V(s, x_s) \ge J(x_s, s, t_f, \mu^*, \nu)\,.$$

This completes the proof. ■

From the above lemma, we see that $V(\tau, x_\tau)$ is a saddle point value, that is, $V(\tau, x_\tau) = J^*(x_s, \tau, t_f)$. Furthermore, it is noted that $V(s, x_s) \ge 0$ for all $s \in [t_0, t_f]$. This can be verified as follows:

From (5), it follows

$$V(s, x_s) = J(x_s, s, t_f, \mu^*, \nu^*) \ge J(x_s, s, t_f, \mu^*, 0)$$

where

$$J(x_s, s, t_f, \mu^*, 0) = \int_s^{t_f} [x^T(\tau)\, Q\, x(\tau) + \mu^{*T}(\tau, x_\tau)\, \mu^*(\tau, x_\tau)]\, d\tau + x^T(t_f)\, F_1\, x(t_f) + \int_{t_f-h}^{t_f} x^T(\tau)\, F_2\, x(\tau)\, d\tau\,.$$

Since $F_1 > 0$ and $F_2 > 0$, we lead to $J(x_s, s, t_f, \mu^*, 0) \ge 0$. Consequently, $V(s, x_s) \ge 0$ for all $s \in [t_0, t_f]$.

Before deriving the receding horizon H_∞ control, we first provide the solution to the finite horizon dynamic game problem in (3). The derivation is based on Lemma 2.1. The procedure taken for derivation of the solution is similar to that used in [5]. We assume the saddle point value has the form

$$V(\tau, x_\tau) = \begin{cases} x^T(\tau) P_1(\tau)\, x(\tau) + 2x^T(\tau) \int_{-h}^{0} P_2(\tau, s)\, x(\tau+s)\, ds \\ + \int_{-h}^{0} \int_{-h}^{0} x^T(\tau+r)\, P_3(\tau, r, s)\, x(\tau+s)\, dr\, ds\,, \\ \qquad\qquad t_0 \le \tau < t_f - h \\ x^T(\tau)\, W_1(\tau)\, x(\tau) + 2x^T(\tau) \int_{-h}^{t_f-\tau-h} W_2(\tau, s)\, x(\tau+s)\, ds \\ + \int_{-h}^{t_f-\tau-h} \int_{-h}^{t_f-\tau-h} x^T(\tau+r)\, W_3(\tau, r, s)\, x(\tau+s)\, dr\, ds \\ + \int_{t_f-h}^{\tau} x^T(\tau)\, F_2\, x(\tau)\, d\tau\,, \\ \qquad\qquad t_f - h \le \tau \le t_f\,. \end{cases}$$

Using the above saddle point value, the saddle point strategies for the dynamic game problem in (3) are given by

$$\mu^*(\tau, x_\tau) = \begin{cases} -B^T\left[P_1(\tau)\,x(\tau) + \int_{-h}^{0} P_2(\tau,s)\,x(\tau+s)\,ds\right], & t_0 \le \tau < t_f - h. \\ -B^T\left[W_1(\tau)\,x(\tau) + \int_{-h}^{t_f-\tau-h} W_2(\tau,s)\,x(\tau+s)\,ds\right], & t_f - h \le \tau \le t_f. \end{cases}$$

and

$$\nu^*(\tau, x_\tau) = \begin{cases} \gamma^{-2}B_w^T\left[P_1(\tau)\,x(\tau) + \int_{-h}^{0} P_2(\tau,s)\,x(\tau+s)\,ds\right], & t_0 \le \tau < t_f - h. \\ \gamma^{-2}B_w^T\left[W_1(\tau)\,x(\tau) + \int_{-h}^{t_f-\tau-h} W_2(\tau,s)\,x(\tau+s)\,ds\right], & t_f - h \le \tau \le t_f. \end{cases}$$

P_1, P_2, and P_3 satisfy the following Riccati-type coupled partial differential equations:

$$\dot{P}_1(\tau) + A^T P_1(\tau) + P_1(\tau)A - P_1(\tau)(BB^T - \gamma^{-2}B_wB_w^T)P_1(\tau) + Q + P_2(\tau,0) + P_2^T(\tau,0) = 0 \tag{6}$$

$$\left(\frac{\partial}{\partial\tau} - \frac{\partial}{\partial s}\right)P_2(\tau,s) + A^T P_2(\tau,s) + P_3(\tau,0,s) - P_1(\tau)\,(BB^T - \gamma^{-2}B_wB_w^T)P_2(\tau,s) = 0 \tag{7}$$

$$\left(\frac{\partial}{\partial\tau} - \frac{\partial}{\partial r} - \frac{\partial}{\partial s}\right)P_3(\tau,r,s) - P_2^T(\tau,r)\,(BB^T - \gamma^{-2}B_wB_w^T)P_2(\tau,s) = 0 \tag{8}$$

with boundary conditions

$$P_2(\tau,-h) = P_1(\tau)A_1 \tag{9}$$

$$P_3(\tau,-h,s) = A_1^T P_2(\tau,s), \tag{10}$$

where $t_0 \le \tau < t_f - h$, $-h \le r \le 0$ and $-h \le s \le 0$. Similarly, W_1, W_2, and W_3 satisfy the following Riccati-type partial differential equations:

$$\dot{W}_1(\tau) + A^T W_1(\tau) + W_1(\tau)A - W_1(\tau)(BB^T - \gamma^{-2}B_wB_w{}^T)\,W_1(\tau) + Q + F_2 = 0$$

$$\left(\frac{\partial}{\partial\tau} - \frac{\partial}{\partial s}\right)W_2(\tau,s) + A^T W_2(\tau,s) - W_1(\tau)(BB^T - \gamma^{-2}B_wB_w^T)\,W_2(\tau,s) = 0$$

$$\left(\frac{\partial}{\partial\tau} - \frac{\partial}{\partial r} - \frac{\partial}{\partial s}\right)W_3(\tau,r,s) - W_2^T(\tau,r)(BB^T - \gamma^{-2}B_wB_w^T)\,W_2(\tau,s) = 0$$

with boundary condition

$$W_2(\tau,-h) = W_1(\tau)A_1$$

$$W_3(\tau,-h,s) = A_1^T W_2(\tau,s),$$

where $t_f - h \le \tau < t_f$, $-h \le r \le 0$ and $-h \le s \le 0$. In addition, P_1, P_2, P_3 and W_1, W_2, W_3 satisfy the following boundary conditions:

$$W_1(t_f) = F_1 \tag{11}$$

$$P_1(t_f - h) = W_1(t_f - h) \tag{12}$$

$$P_2(t_f - h, s) = W_2(t_f - h, s) \tag{13}$$

$$P_3(t_f - h, r, s) = W_3(t_f - h, r, s). \tag{14}$$

P_1, P_2, P_3, and W_1, W_2, W_3 are solved backward in time from t_f to t_0. Because the system is time-invariant, the shape of P_1, P_2, P_3, and W_1, W_2, W_3 are only characterized by the difference between the initial time and the final time, that is, $t_f - t_0$. If $t_f - t_0$ varies, the values of P_1, P_2, P_3, and W_1, W_2, W_3 at the initial time, t_0, also vary. However, if $t_f - t_0$ is fixed to a constant value, the values are all the same at the initial time. For example, $P_1(t_0)$ with $t_0 = 1$ and $t_f = 5$ is equal to $P_1(t_0)$ with $t_0 = 2$ and $t_f = 6$. If we take receding horizon strategy, t_0 and t_f corresponds to t and $t + T_p$, respectively, where t denotes the current time. The difference between the initial time and the terminal time is always constant to be T_p. Therefore, $P_1(t_0)$ reduces to a constant matrix regardless of the value of t_0. Let's introduce new notations as follows:

$$\bar{P}_1 \triangleq P_1(t_0), \quad \bar{P}_2(s) \triangleq P_2(t_0, s),$$

$$\bar{W}_1 \triangleq W_1(t_0), \quad \bar{W}_2(s) \triangleq W_2(t_0, s).$$

Finally, the receding horizon H_∞ control is represented as a distributed state feedback strategy as follows:

$$u_R(x_t) = \begin{cases} -B^T\left[\bar{P}_1\,x(t) + \int_{-h}^{0} \bar{P}_2(s)\,x(t+s)\,ds\right], & T_p > h \\ -B^T\left[\bar{W}_1\,x(t) + \int_{-h}^{T_p-h} \bar{W}_2(s)\,x(t+s)\,ds\right], & 0 < T_p \le h. \end{cases} \tag{15}$$

It is noted that the feedback strategy needs only the state trajectories for time interval $[t-h, t]$ and is invariant with time.

Remark 2.2. In order to solve Riccati-type coupled partial differential equations given above, we utilize a numerical algorithm presented in [14]. The main idea of the method in [14] is that the original coupled partial differential equations can be transformed into coupled ordinary differential

equations by appropriate change of variables. The coupled differential equations are solved simultaneously backward in time. The well-known fourth-order Runge-Kutta method can be used to solve the differential equations for the time interval $[t_f - h, t_f]$. However, Euler's method is used for the time interval $[t_f, t_f - h)$. Therefore the solution corresponding to the time interval $[t_f, t_f - h)$ is liable to bigger numerical error. By decreasing the numerical integration step size, more accurate numerical solution is obtained. However, this causes more computational effort. That is, there exists a trade-off between the accuracy of the numerical solution and the required computational effort.

We have constructed RHHC from the solution to a finite horizon dynamic game problem. However, the only thing we can say about the control is that it is obtained based on the receding horizon strategy. Nothing can be said about the asymptotic stability and H_∞-norm boundedness yet. We therefore will investigate those issues in the next two sections.

III. NONINCREASING MONOTONICITY OF A SADDLE POINT VALUE

Nonincreasing monotonicity of the saddle point value plays an important role in proving the closed-loop stability and H_∞-norm bound for delay-free systems. As will be shown later, this is also the case with time-delay systems. In what follows, we will show how to choose terminal weighting matrices such that the saddle point value satisfies the nonincreasing monotonicity.

Theorem 3.1. Given $\gamma > 0$, assume that there exist $X > 0$, S, Y, and Y_1 such that

$$\begin{bmatrix} (AX+BY)+(AX+BY)^T & A_1S+BY_1 & B_w & X\,Q^{\frac{1}{2}} & Y^T & X \\ \star & -S & 0 & 0 & Y_1^T & 0 \\ \star & \star & -\gamma^2 I & 0 & 0 & 0 \\ \star & \star & \star & -I & 0 & 0 \\ \star & \star & \star & \star & -I & 0 \\ \star & \star & \star & \star & \star & -S \end{bmatrix} \le 0 . \tag{16}$$

If we choose terminal weighting matrices F_1 and F_2 such that $F_1 = X^{-1}$ and $F_2 = S^{-1}$, the saddle point value $J^*(x_{t_0}, t_0, \sigma)$ satisfies the following nonincreasing monotonicity property:

$$\frac{\partial J^*(x_{t_0}, t_0, \sigma)}{\partial \sigma} \le 0 , \quad \forall \sigma > t_0 . \tag{17}$$

Proof.

$$\begin{aligned}
&\frac{\partial J^*(x_{t_0}, t_0, \sigma)}{\partial \sigma} \\
&= \frac{1}{\Delta}\Big\{\int_{t_0}^{\sigma+\Delta}[\bar{x}^T(\tau)\,Q\,\bar{x}(\tau)+\bar{\mu}^T(\tau,\bar{x}_\tau)\,\mu(\tau,\bar{x}_\tau) \\
&\quad -\gamma^2\bar{\nu}^T(\tau,\bar{x}_\tau)\,\bar{\nu}(\tau,\bar{x}_\tau)]\,d\tau \\
&\quad +\bar{x}^T(\sigma+\Delta)\,F_1\,\bar{x}(\sigma+\Delta) \\
&\quad +\int_{\sigma+\Delta-h}^{\sigma+\Delta}\bar{x}^T(\tau)\,F_2\,\bar{x}(\tau)\,d\tau \\
&\quad -\int_{t_0}^{\sigma}[\hat{x}^T(\tau)\,Q\,\hat{x}(\tau)+\hat{\mu}^T(\tau,\hat{x}_\tau)\,\hat{\mu}(\tau,\hat{x}_\tau) \\
&\quad -\gamma^2\,\hat{\nu}^T(\tau,\hat{x}_\tau)\,\hat{\nu}(\tau,\hat{x}_\tau)]\,d\tau \\
&\quad +\hat{x}^T(\sigma)\,F_1\,\hat{x}(\sigma)-\int_{\sigma-h}^{\sigma}\hat{x}^T(\tau)\,F_2\,\hat{x}(\tau)\,d\tau\Big\}
\end{aligned}$$

where the pair $(\bar{\mu},\bar{\nu})$ is a saddle point solution for $J(x_{t_0}, t_0, \sigma + \Delta, u, w)$ and the pair $(\hat{\mu},\hat{\nu})$ is a saddle point solution for $J(x_{t_0}, t_0, \sigma, u, w)$. $\bar{x}$ denotes the state trajectory resulting from the strategies $\bar{\mu}$ and $\bar{\nu}$ and $\hat{x}$ denotes the state trajectory resulting from the strategies $\hat{\mu}$ and $\hat{\nu}$. Replace the strategy $\bar{\mu}$ and $\hat{\nu}$ by $\hat{\mu}$ and $\bar{\nu}$ up to σ and use $u(\tau) = Kx(\tau) + K_1x(\tau - h)$ and $w(\tau) = \hat{\nu}(\tau, x_\tau)$ for $\tau \ge \sigma$. It is noted that, since we have changed strategies, the resulting state trajectory is neither $\bar{x}$ nor $\hat{x}$. Let's denote the resulting state trajectory by x. Then we obtain

$$\begin{aligned}
&\frac{\partial J^*(x_{t_0}, t_0, \sigma)}{\partial \sigma} \\
&\le \lim_{\Delta\to 0}\frac{1}{\Delta}\Big\{\int_{\sigma}^{\sigma+\Delta}[x^T(\tau)\,Q\,x(\tau) \\
&\quad +[K\,x(\tau)+K_1\,x(\tau-h)]^T[K\,x(\tau)+K_1\,x(\tau-h)] \\
&\quad -\gamma^2 w^T(\tau)\,w(\tau)]\,d\tau \\
&\quad +x^T(\sigma+\Delta)\,F_1\,x(\sigma+\Delta) \\
&\quad +\int_{\sigma+\Delta-h}^{\sigma+\Delta}x^T(\tau)\,F_2\,x(\tau)\,d\tau \\
&\quad -x^T(\sigma)\,F_1\,x(\sigma)-\int_{\sigma-h}^{\sigma}x^T(\tau)\,F_2\,x(\tau)\,d\tau\Big\} \\
&= x^T(\sigma)\,Q\,x(\sigma) \\
&\quad +[K\,x(\sigma)+K_1\,x(\sigma-h)]^T[K\,x(\sigma)+K_1\,x(\sigma-h)] \\
&\quad -\gamma^2 w^T(\tau)\,w(\tau) \\
&\quad +\frac{d}{d\sigma}\Big\{x^T(\sigma)\,F_1\,x(\sigma)+\int_{\sigma-h}^{\sigma}x^T(\tau)F_2\,x(\tau)\,d\tau\Big\} \\
&= x^T(\sigma)\,Q\,x(\sigma) \\
&\quad +[K\,x(\sigma)+K_1\,x(\sigma-h)]^T[K\,x(\sigma)+K_1\,x(\sigma-h)] \\
&\quad -\gamma^2 w^T(\tau)\,w(\tau)+2\dot{x}^T(\sigma)\,F_1\,x(\sigma) \\
&\quad +x^T(\sigma)\,F_2\,x(\sigma)-x^T(\sigma-h)\,F_2\,x(\sigma-h) .
\end{aligned}$$

After substituting $\dot{x}(\sigma) = (A+BK)\,x(\sigma) + (A_1 + BK_1) \cdot x(\sigma - h) + B_w w(\sigma)$ into the above, we obtain

$$\frac{\partial J^*(x_{t_0}, t_0, \sigma)}{\partial \sigma} \le \begin{bmatrix} x(\sigma) \\ x(\sigma-h) \\ w(\sigma) \end{bmatrix}^T \underbrace{\begin{bmatrix} \Lambda_{11} & F_1(A_1+BK_1)+K^TK_1 & F_1B_w \\ \star & -F_2+K_1^TK_1 & 0 \\ \star & \star & -\gamma^2 I \end{bmatrix}}_{\Lambda} \begin{bmatrix} x(\sigma) \\ x(\sigma-h) \\ w(\sigma) \end{bmatrix}$$

where

$$\Lambda_{11} = (A+BK)^T F_1 + F_1(A+BK) + Q + K^TK + F_2.$$

It is apparent that, if $\Lambda \le 0$, nonincreasing monotonicity in (17) holds. $\Lambda \le 0$ can be rewritten as follows:

$$\begin{bmatrix} (A+BK)^T F_1 + F_1(A+BK) & F_1(A+BK_1) \\ \star & -F_2 \end{bmatrix} + \begin{bmatrix} Q^{1/2} & 0 \\ K & K_1 \\ I & 0 \end{bmatrix}^T \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & F_2^{-1} \end{bmatrix}^{-1} \begin{bmatrix} Q^{1/2} & 0 \\ K & K_1 \\ I & 0 \end{bmatrix} \le 0.$$

Pre- and post-multiply the above matrix inequality by $diag\{F_1^{-1}, F_2^{-1}\}$. The above inequality is then equivalently changed into (16) according to the Schur complement. This completes the proof. ■

The nonincreasing monotonicity of the saddle point value implies that the saddle point value does not increase even though we increase the horizon length. As will be shown in the next section, this property plays an important role in RHHC's achieving closed-loop stability and H_∞-norm boundedness.

IV. ASYMPTOTIC STABILITY AND H_∞-NORM BOUNDEDNESS

In this section, we show that the proposed receding horizon control achieves the closed-loop asymptotic stability for zero disturbance and satisfies the H_∞-norm boundedness for zero initial condition.

Theorem 4.1. Given $Q > 0$ and $\gamma > 0$, if $\frac{J^*(x_{t_0}, t_0, \sigma)}{\partial \sigma} \le 0$ for $\sigma > t_0$, the system (1) controlled by the RHHC in (15) is asymptotically stable for zero disturbance and satisfies infinite horizon H_∞-norm bound for zero initial condition.

Proof. For any $\theta > 0$, the saddle point value V^* satisfies

$$\begin{aligned} &J^*(x_t, t, t+T_p) \\ &= \int_t^{t+\theta} [x^T(t)\,Q\,x(t) + \mu^{*T}(\tau, x_\tau)\,\mu^*(\tau, x_\tau) \\ &\qquad - \gamma^2\,\nu^{*T}(\tau, x_\tau)\,\nu^*(\tau, x_\tau)]\,d\tau \\ &\quad + J^*(x_{t+\theta}, t+\theta, t+T_p), \end{aligned}$$

where the pair (μ^*, ν^*) is a saddle point solution for $J(x_t, t, t+T_p)$. Replace the saddle point strategy $\nu^*(\tau, x_\tau)$ by $\nu(\tau, x_\tau)$ given by

$$\nu(\tau, x_\tau) = \begin{cases} w(\tau) & , \quad t \le \tau < t+\theta \\ \nu^*(\tau, x_\tau) & , \quad t+\theta \le \tau \le t+T_p \end{cases}$$

where $w(\tau)$ denotes an arbitrary signal. Recalling $J^*(x_t, t, t+T_p, u, w) = J(x_t, t, t+T_p, \mu^*, \nu^*)$, we obtain

$$\begin{aligned} &J^*(x_t, t, t+T_p) \ge J(x_t, t, t+T_p, \mu^*, \nu) \\ &= \int_t^{t+\theta} [\bar{x}^T(\tau)\,Q\,\bar{x}(\tau) + \mu^T(\tau, \bar{x}_\tau)\,\mu(\tau, \bar{x}_\tau) \\ &\qquad - \gamma^2 w^T(\tau)\,w(\tau)]\,d\tau \\ &\quad + J(\bar{x}_{t+\theta}, t+\theta, t+T_p, \mu^*, \nu^*) \end{aligned}$$

where $\bar{x}$ denotes the state trajectory resulting from the strategy μ^* and ν. It is noted that the strategy pair (μ^*, ν^*) is the saddle point solution not only for $J(x_t, t, t+T_p, u, w)$ but also for $J(x_{t+\theta}, t+\theta, t+T_p, u, w)$ because the terminal time is $t + T_p$ for both cases. Therefore $J(\bar{x}_{t+\theta}, t+\theta, t+T_p, \mu^*, \nu^*) = J^*(\bar{x}_{t+\theta}, t+\theta, t+T_p)$. Furthermore, from the fact that $\frac{J^*(x_{t_0}, t_0, \sigma)}{\partial \sigma} \le 0$ for $\sigma > t_0$, it follows $J^*(\bar{x}_{t+\theta}, t+\theta, t+T_p) \ge J^*(\bar{x}_{t+\theta}, t+\theta, t+T_p+\theta)$. This, in turn, leads to

$$\begin{aligned} &J^*(x_t,\ t, t+T_p) \\ &\ge \int_t^{t+\theta} [\bar{x}^T(\tau)\,Q\,\bar{x}(\tau) + \mu^{*T}(\tau, \bar{x}_\tau)\,\mu^*(\tau, \bar{x}_\tau) \\ &\quad - \gamma^2 w^T(\tau)\,w(\tau)]\,d\tau + J^*(\bar{x}_{t+\theta},\ t+\theta,\ t+T_p+\theta). \end{aligned}$$

Therefore we obtain

$$\begin{aligned} &\frac{J^*(\bar{x}_{t+\theta}, t+\theta, t+T_p+\theta) - J^*(x_t, t, t+T_p)}{\theta} \\ &\le -\frac{1}{\theta}\int_t^{t+\theta} \Big[\bar{x}^T(\tau)\,Q\,\bar{x}(\tau) + \mu^{*T}(\tau, \bar{x}_\tau)\,\mu^*(\tau, \bar{x}_\tau) \\ &\qquad - \gamma^2 w^T(\tau)\,w(\tau)\Big]\,d\tau \end{aligned}$$

When $\theta \to 0$, we have

$$\begin{aligned} &\frac{dJ^*(x_t, t, t+T_p)}{dt} \\ &\le -[x^T(t)\,Q\,x(t) + u_R^T(x_t)\,u_R(x_t) - \gamma^2 w^T(t)\,w(t)] \end{aligned} \tag{18}$$

$$= -\psi^T \begin{bmatrix} Q + \bar{P}_1 BB^T \bar{P}_1 & \bar{P}_1 B \\ B^T \bar{P}_1 & I \end{bmatrix} \psi - \gamma^2 w^T(t)\, w(t) \qquad (19)$$

where

$$\psi = \begin{bmatrix} x(t) \\ B^T \int_{-h}^{0} \bar{P}_2(s)\, x(t+s)\, ds \end{bmatrix}.$$

In case of $w(t) = 0$, we know that

$$\frac{dJ^*(t,t,t+T_p)}{dt} \le -\psi^T \begin{bmatrix} Q + \bar{P}_1 BB^T \bar{P}_1 & \bar{P}_1 B \\ B^T \bar{P}_1 & I \end{bmatrix} \psi$$

$$\le -x^T(t)\, Q\, x(t)$$

$$\le -\lambda_{min}(Q)\, \| x(t) \|^2 \quad .$$

We can conclude that $J^*(x_t, t, t + T_p)$ is a Lyapunov-Krasovskii functional from the above derivation. Therefore, the closed-loop system is asymptotically stable. We then prove that H_∞-norm bound is guaranteed for the closed-loop system. Consider an infinite horizon cost J_w given by

$$J_w = \int_0^\infty [x^T(t)\, Q\, x(t) + u_R^T(x_t)\, u_R(x_t) - \gamma^2 w^T(t)\, w(t)]\, dt\,.$$

We have only to show that $J_w \le 0$ for the proof of H_∞-norm bound.

$$J_w = \int_0^\infty [x^T(t)\, Q\, x(t) + u_R^T(x_t)\, u_R(x_t) - \gamma^2 w^T(t)\, w(t) + \frac{d}{dt} J^*(x_t, t, t+T_p)]\, dt + J^*(x_0, 0, T_p) - J^*(x_t, t, t+T_p)|_{t=\infty}$$

For zero initial condition, *i.e.* $x(s) = 0$, $s \in [-h, 0]$, $J^*(x_0, 0, T_p)$ is equal to zero. Furthermore, since the saddle point value is nonnegative-definite, we have $J^*(x_t, t, t + T_p)|_{t=\infty} \ge 0$. This, in turn, leads to

$$J_w \le \int_0^\infty [x^T(t)\, Q\, x(t) + u_R^T(x_t)\, u_R(x_t) - \gamma^2 w^T(t)\, w(t) + \frac{d}{dt} J^*(x_t, t, t+T_p)]\, dt$$

The integrand above is less than or equal to zero from (18). Consequently we can conclude that $J_w \le 0$. This completes the proof. ■

Theorem 4.1 states that the nonincreasing monotonicity of the saddle point value is the sufficient condition for the closed-loop stability and the H_∞-norm boundedness. An LMI condition on the terminal weighting matrices under which the saddle point satisfies nonincreasing monotonicity was given in Theorem 3.1. Therefore, we lead to the following corollary:

Corollary 4.1. Given $Q > 0$ and $\gamma > 0$, if the LMI (16) is feasible and we can obtain two terminal weighting matrices F_1 and F_2, the system (1) controlled by the proposed RHHC in (15) is asymptotically stable for zero disturbance and satisfies infinite horizon H_∞ norm bound for zero initial condition.

V. NUMERICAL EXAMPLE

In this section, we provide a numerical example in order to illustrate the properties of the proposed RHHC. We use a computer with Pantium 4 CPU (2.8GHz, 496MByte RAM) for computation and simulation. Consider a chemical reactor system taken from [15]. The system matrices are given by

$$A = \begin{bmatrix} -4.93 & -1.01 & 0 & 0 \\ -3.20 & -5.30 & -12.8 & 0 \\ -6.40 & 0.347 & -32.5 & -1.04 \\ 0 & 0.833 & 11.0 & -3.96 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 1.92 & 0 & 0 & 0 \\ 0 & 1.92 & 0 & 0 \\ 0 & 0 & 1.87 & 0 \\ 0 & 0 & 0 & 0.724 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad B_w = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

and the delay length is $h = 1$. We chose $Q = I$ and $\gamma = 0.85$. We obtained terminal weighting matrices F_1 and F_2 by solving an LMI (16) and using the relation $F_1 = X^{-1}$, $F_2 = S^{-1}$ as follows:

$$F_1 = \begin{bmatrix} 1.8193 & -0.6531 & 0.2726 & -0.0417 \\ \star & 0.3979 & -0.1397 & 0.0605 \\ \star & \star & 0.1042 & 0.0467 \\ \star & \star & \star & 0.2169 \end{bmatrix}$$

$$F_2 = \begin{bmatrix} 3.2895 & -1.3429 & 0.5725 & 0.0081 \\ \star & 0.6921 & -0.2584 & 0.0258 \\ \star & \star & 0.1380 & 0.0301 \\ \star & \star & \star & 0.0466 \end{bmatrix}$$

Y and Y_1 are obtained as follows:

$$Y = \begin{bmatrix} -1.0239 & 0.0628 & 0.2685 & -0.0725 \\ -0.0091 & -0.9677 & 0.1219 & -0.0353 \end{bmatrix}$$

$$Y_1 = \begin{bmatrix} 0.0023 & 0.0049 & -0.0031 & -0.0037 \\ 0.0008 & 0.0000 & -0.0055 & 0.0028 \end{bmatrix}$$

For the prediction horizon length T_p = 1.5, we obtained $\bar{P}_1$ and $\bar{P}_2(s)$ in (15) from the solution to Riccati-type partial differential equations given in Section 2. $\bar{P}_1$ is given below and the trajectory of (1, 1) element of $P_2(s)$ is shown for $-h \le s \le 0$ in Fig. 1. The remaining trajectories of $P_2(s)$ are omitted due to the space limitation.

$$\bar{P}_1 = \begin{bmatrix} 0.6737 & -0.2608 & 0.1159 & 0.0007 \\ \star & 0.1860 & -0.0643 & 0.0200 \\ \star & \star & 0.0517 & 0.0283 \\ \star & \star & \star & 0.1213 \end{bmatrix}$$

Numerical integration step size was taken to be 0.02 seconds. The total time taken to solve the partial differential equations turned out to be 38.34 seconds. This denotes that relatively big computation is required to obtain the final control law given in (15).

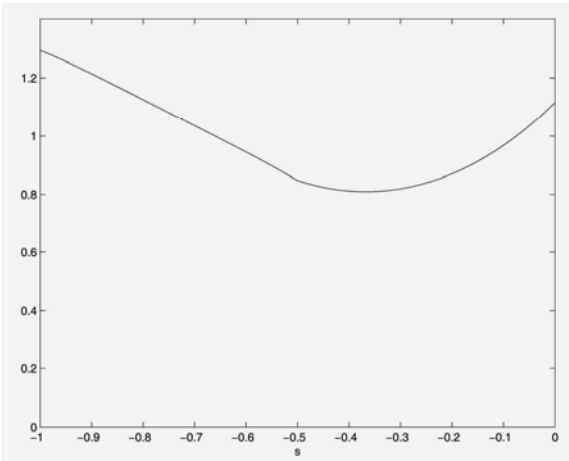

Fig. 1. The trajectory of (1, 1) element of $P_2(s)$ for $-h \le s \le 0$.

In order to illustrate the stability and the H_∞-norm boundedness, we applied the disturbance input $w(t)$ that has the shape in Figs. 2 and 3 shows the state response of the closed-loop system to the disturbance in Fig. 2. It clearly shows that the resulting closed-loop system is stable. The value of $\|z\|_2 / \|w\|_2$ was computed to be 0.5384, which is less than γ = 0.85. This supports that the closed-loop system satisfies H_∞-norm boundedness. It is noted that the total time taken to do 15-second-long simulation is about 0.8 seconds. Therefore, the proposed controller can be implemented real-time if high speed processors, such as DSP, are used.

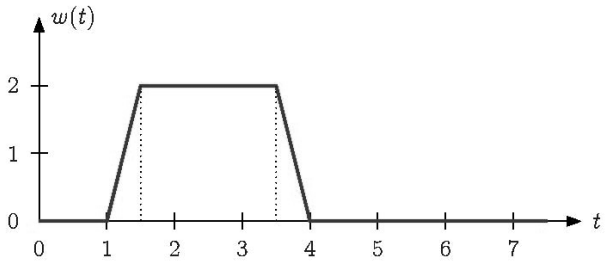

Fig. 2. Disturbance, $w(t)$.

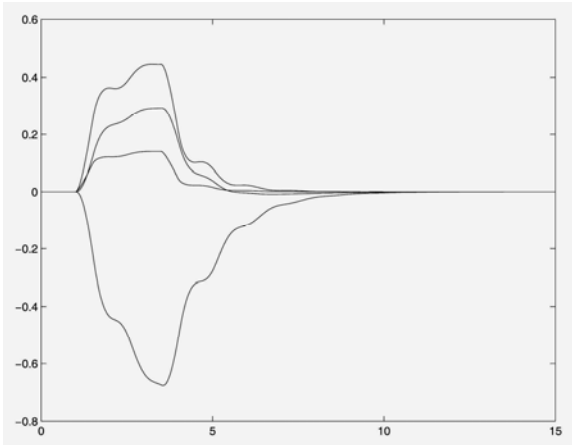

Fig. 3. State response to the disturbance input.

VI. CONCLUSION

In this paper, we proposed the receding horizon H_∞ control (RHHC) for linear systems with a state-delay. Firstly, we proposed a new cost function for a dynamic game problem. The terminal weighting term is parameterized by two terminal weighting matrices. Secondly, we derived a saddle point solution to a finite horizon dynamic game problem. Thirdly, the receding horizon H_∞ control was constructed from the obtained saddle point solution. We showed that, under the nonincreasing monotonicity condition of a saddle point value, the proposed receding horizon H_∞ control is stabilizing and satisfies the H_∞-norm bound. We proposed an LMI condition on the terminal weighting matrices, under which the saddle point value satisfies the nonincreasinn monotonicity. Main contribution of this work is that it extended the receding horizon H_∞ control for delay-free systems to that for time-delay systems for the first time.

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