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Brief paper

Delay-dependent robust H_∞ control for uncertain systems with a state-delay[☆]

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Abstract

A robust H_∞ control for uncertain linear systems with a state-delay is described. Systems with norm-bounded parameter uncertainties are considered and linear memoryless state feedback controllers are obtained. Firstly, a delay-dependent bounded real lemma for systems with a state-delay is presented in terms of linear matrix inequalities (LMIs). By taking a new Lyapunov–Krasovsii functional, neither model transformation nor bounding for cross terms is required to obtain delay-dependent results. Secondly, based on the bounded real lemma obtained, delay-dependent condition for the existence of robust H_∞ control is presented in terms of nonlinear matrix inequalities. In order to solve these nonlinear matrix inequalities, an iterative algorithm involving convex optimization is proposed. Numerical examples show that the proposed methods are much less conservative than existing results.

Keywords: Time-delay systems; Delay-dependent; Bounded real lemma; H_∞ control; Uncertain systems

1. Introduction

There have been considerable research efforts on robust H_∞ control for uncertain time-delay systems, in particular, with parameter uncertainties. The existing results for H_∞ control of time-delay systems deal with either one of two types of stabilization: delay-independent stabilization (Choi & Chung, 1997; Kapila & Haddad, 1998; Zribi & Mahmoud, 1999; Kim & Park, 1999) and delay-dependent stabilization (de Souza & Li, 1999; Fridman & Shaked, 2001; Lee, Moon, & Kwon, 2001; Fridman & Shaked, 2002). Recent research effort is focused more on delay-dependent stabilization. The main objective of the delay-dependent H_∞ control is to obtain a controller that allows a maximum delay size for a fixed H_∞ performance bound or achieves a minimum H_∞ performance bound for a fixed delay size. The conservatism in the delay-dependent H_∞ control is hence measured by the allowable delay size or performance bound obtained.

In the past few years, there have been various approaches to reduce the conservatism of delay-dependent conditions by using new bounding for cross terms or choosing new Lyapunov–Krasovskii functional. The delay-dependent stability criterion of Park, Moon, and Kwon (1998) and Park (1999) is based on a so-called *Park's inequality* for bounding cross terms. This stability criterion was later extended to controller synthesis (Moon, 1998). However, major drawback in using the bounding of Park et al. (1998) and Park (1999) is that some matrix variables should be limited to a certain structure to obtain controller synthesis conditions in terms of LMIs. This limitation introduces some conservatism. In Moon, Park, Kwon, and Lee (2001) a new inequality, which is more general than the Park's inequality, was introduced for bounding cross terms and controller synthesis conditions were presented in terms of nonlinear matrix inequalities in order to reduce the conservatism. An iterative algorithm was developed to solve the nonlinear matrix inequalities (Moon et al., 2001). Stabilization method of Moon et al. (2001) was later extended to H_∞ control

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in Lee et al. (2001). Recently, it was shown that conservatism can be further reduced by taking a new Lyapunov–Krasovskii functional (Fridman & Shaked, 2002). The results use the inequalities of Park (1999) for bounding cross terms. However, conservatism still remains because some of matrix variables have limited structure to obtain controller synthesis condition in terms of LMIs.

This paper is based on a new Lyapunov–Krasovskii functional. It is well-known that existing delay-dependent methods commonly require some kind of model transformation and bounding for cross terms. However, in the present paper, those are unnecessary thanks to a newly chosen Lyapunov–Krasovskii functional. A less conservative delay-dependent robust H_∞ control is proposed for uncertain linear systems with a state-delay and parameter uncertainties based on the new Lyapunov–Krasovskii functional. We focus on memoryless state-feedback control. As an initial step, a new delay-dependent bounded real lemma for systems with a state-delay is derived in terms of LMIs. Based on the bounded real lemma obtained, a condition for the existence of a robust H_∞ control is given in terms of nonlinear matrix inequalities. The reason for presenting controller synthesis condition in terms of nonlinear matrix inequalities is because much conservatism can be reduced by doing so. Because these nonlinear matrix inequalities lead to a nonconvex feasibility problem, an iterative algorithm involving convex optimization, which is similar to the one in Moon et al. (2001), is given to solve the problem.

2. Problem statement and preliminaries

Consider the following uncertain systems with a single state-delay:

$$\dot{x}(t) = [A + F\Delta(t)E]\,x(t) + [A_1 + F\Delta(t)E_1]\,x(t-h) + [B + F\Delta(t)E_b]u(t) + B_w w(t), \quad (1)$$

$$z(t) = \begin{bmatrix} Cx(t) + D_w w(t) \\ C_1 x(t-h) \\ Du(t) \end{bmatrix},$$

$$x(t) = 0, \quad t \in [-\bar{h}, 0],$$

where $x(t) \in \mathbf{R}^n$ is the state, $u(t) \in \mathbf{R}^m$ is the control input, $w(t) \in \mathbf{R}^p$ is the disturbance input that belongs to $L_2[0,\infty)$, $z(t) \in \mathbf{R}^q$ is the controlled output, $h \geqslant 0$ is an unknown but constant delay, and $\bar{h}$ is a constant satisfying $h \leqslant \bar{h}$. A, A_1, B, B_w, C, C_1, D, D_w, F, E, E_1, and E_b are known real constant matrices of appropriate dimensions and $\Delta(t)$ denotes time-varying parameter uncertainties. $\Delta(t)$ is assumed to be of diagonal form

$$\Delta(t) = \mathrm{diag}\{\Delta_1(t), \ldots, \Delta_r(t)\},$$

where $\Delta_i(t) \in \mathbf{R}^{p_i \times q_i}$, $i = 1, \ldots, r$, are unknown real time-varying matrices satisfying

$$\Delta_i^{\mathrm{T}}(t)\Delta_i(t) \leqslant I \quad \forall t \geqslant 0.$$

Throughout the paper, I denotes an identity matrix of appropriate dimension.

We are interested in designing a memoryless state-feedback controller

$$u(t) = Kx(t), \quad (2)$$

where $K \in \mathbf{R}^{m\times n}$ is a constant matrix. The aim of this paper is to develop a delay-dependent robust H_∞ control such that, for all admissible uncertainties and any constant time-delay h satisfying $0 \leqslant h \leqslant \bar{h}$,

(1) the closed-loop system is stable;
(2) the closed-loop system guarantees, under zero initial condition,

$$\|z(t)\|_2 < \gamma \|w(t)\|_2$$

for all nonzero $w \in L_2[0,\infty)$ and some prescribed constant $\gamma > 0$.

The following lemma is introduced in order to handle the parameter uncertainties:

Lemma 2.1. *Let F, E, and Δ be real matrices of appropriate dimensions with $\Delta = \mathrm{diag}\{\Delta_1, \ldots, \Delta_r\}$, $\Delta_i^{\mathrm{T}}\Delta_i \leqslant I$, $i = 1, \ldots, r$. Then, for any real matrix $\Lambda = \mathrm{diag}\{\lambda_1 I, \ldots, \lambda_r I\} > 0$, the following inequality holds*:

$$F\Delta E + E^{\mathrm{T}}\Delta^{\mathrm{T}}F^{\mathrm{T}} \leqslant F\Lambda F^{\mathrm{T}} + E^{\mathrm{T}}\Lambda^{-1}E. \quad (3)$$

Proof. The basic idea for the proof is to use the fact $(\Lambda^{\frac{1}{2}}F^{\mathrm{T}} - \Lambda^{-\frac{1}{2}}\Delta E)^{\mathrm{T}}(\Lambda^{\frac{1}{2}}F^{\mathrm{T}} - \Lambda^{-\frac{1}{2}}\Delta E) \geqslant 0$. □

Remark 2.1. In (1), the same matrix $\Delta(t)$ appears in the perturbation of all the system matrices. Some papers use independent perturbations on e.g. A and A_1 as follows (de Souza & Li, 1999; Moon et al., 2001):

$$\dot{x}(t) = [A + \bar{F}\bar{\Delta}(t)\bar{E}]x(t) + [A_1 + \bar{F}_1\bar{\Delta}_1(t)\bar{E}_1]\,x(t-h) + [B + \bar{F}\bar{\Delta}(t)\bar{E}_b]u(t). \quad (4)$$

Let us take

$$F = [\bar{F} \;\; \bar{F}_1], \quad \Delta(t) = \begin{bmatrix} \bar{\Delta}(t) & 0 \\ 0 & \bar{\Delta}_1(t) \end{bmatrix},$$

$$E = \begin{bmatrix} \bar{E} \\ 0 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0 \\ \bar{E}_1 \end{bmatrix}, \quad E_b = \begin{bmatrix} \bar{E}_b \\ 0 \end{bmatrix}.$$

This shows that, with an appropriate structure of E, E_1, E_b, and Δ, (1) can always deal with independent perturbations in (4).

3. Delay-dependent bounded real lemma

Let us consider nominal systems with a state-delay

$$\dot{x}(t) = Ax(t) + A_1x(t-h) + B_ww(t),$$
$$z(t) = \begin{bmatrix} Cx(t) + D_ww(t) \\ C_1x(t-h) \end{bmatrix},$$
$$x(t) = 0, \quad t \in [-\bar{h}, 0]. \tag{5}$$

For some prescribed value $\gamma > 0$, we denote the H_∞ norm boundedness of the transfer function from w to z, T_{zw}, by $\|T_{zw}\|_\infty < \gamma$. However, throughout this paper, we prefer the notation $\|z(t)\|_2 < \gamma\|w(t)\|_2$, $\forall w(t) \in L_2[0,\infty)$ because the H_∞ norm is equal to the induced 2-norm in the time domain. For delay-free systems, the condition for $\|T_{zw}\|_\infty < \gamma$ is stated in the well-known *bounded real lemma* (BRL), which is a necessary and sufficient condition. In the following theorem, we present a new bounded real lemma for systems with a state-delay. However, unlike the BRL for delay-free systems, the proposed BRL gives only a sufficient condition.

Theorem 3.1. *For some prescribed $\gamma > 0$ and $\bar{h} \geqslant 0$, assume that there exist $P_1 > 0$, P_2, P_3, Q, X_{11}, X_{12}, X_{22}, Y_1, Y_2, and $Z > 0$ such that*

$$\begin{bmatrix} \Gamma_{11} & \Gamma_{12} & P_2^{\mathrm{T}}A_1 - Y_1 & P_2^{\mathrm{T}}B_w & C^{\mathrm{T}} & 0 \\ \star & \Gamma_{22} & P_3^{\mathrm{T}}A_1 - Y_2 & P_3^{\mathrm{T}}B_w & 0 & 0 \\ \star & \star & -Q & 0 & 0 & C_1^{\mathrm{T}} \\ \star & \star & \star & -\gamma^2 I & D_w^{\mathrm{T}} & 0 \\ \star & \star & \star & \star & -I & 0 \\ \star & \star & \star & \star & \star & -I \end{bmatrix} < 0, \tag{6}$$

$$\begin{bmatrix} X_{11} & X_{12} & Y_1 \\ \star & X_{22} & Y_2 \\ \star & \star & Z \end{bmatrix} \geqslant 0, \tag{7}$$

where

$$\Gamma_{11} \triangleq P_2^{\mathrm{T}}A + A^{\mathrm{T}}P_2 + \bar{h}X_{11} + Q + Y_1 + Y_1^{\mathrm{T}},$$
$$\Gamma_{12} \triangleq P_1 - P_2^{\mathrm{T}} + A^{\mathrm{T}}P_3 + \bar{h}X_{12} + Y_2^{\mathrm{T}},$$
$$\Gamma_{22} \triangleq -P_3 - P_3^{\mathrm{T}} + \bar{h}X_{22} + \bar{h}Z.$$

Then system (5) *is stable and satisfies $\|z\|_2 < \gamma\|w\|_2$ for all nonzero $w \in L_2[0,\infty)$ and any constant time-delay h satisfying $0 \leqslant h \leqslant \bar{h}$.*

Proof. Consider the following index:

$$J_{zw} = \int_0^\infty [z^{\mathrm{T}}(t)z(t) - \gamma^2 w^{\mathrm{T}}(t)w(t)]\,\mathrm{d}t.$$

Let us choose a Lyapunov–Krasovskii functional candidate $V(x_t)$ as follows:

$$V(x_t) = V_1(x_t) + V_2(x_t) + V_3(x_t) + V_4(x_t), \tag{8}$$

where

$$V_1(x_t) \triangleq \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^{\mathrm{T}} HP \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix},$$
$$V_2(x_t) \triangleq \int_{-h}^{0}\int_{t+\beta}^{t} \dot{x}^{\mathrm{T}}(\alpha)Z\dot{x}(\alpha)\,\mathrm{d}\alpha\,\mathrm{d}\beta,$$
$$V_3(x_t) \triangleq \int_{t-h}^{t} x^{\mathrm{T}}(\alpha)Qx(\alpha)\,\mathrm{d}\alpha,$$
$$V_4(x_t) \triangleq \int_0^t\int_{\beta-h}^{\beta} \begin{bmatrix} x(\beta) \\ \dot{x}(\beta) \\ \dot{x}(\alpha) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} X_{11} & X_{12} & Y_1 \\ \star & X_{22} & Y_2 \\ \star & \star & Z \end{bmatrix} \times \begin{bmatrix} x(\beta) \\ \dot{x}(\beta) \\ \dot{x}(\alpha) \end{bmatrix} \mathrm{d}\alpha\,\mathrm{d}\beta,$$
$$P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix}, \quad H = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},$$

and

$$P_1 > 0, Z > 0, Q > 0, \begin{bmatrix} X_{11} & X_{12} & Y_1 \\ \star & X_{22} & Y_2 \\ \star & \star & Z \end{bmatrix} \geqslant 0.$$

x_t denotes $x(t+\theta)$ for $-h \leqslant \theta \leqslant 0$. Note that $V(x_t) > 0$ unless $\|x_t\|_c = 0$, where

$$\|x_t\|_c = \max_\theta \|x(\theta)\|_2, \quad -h \leqslant \theta \leqslant 0.$$

The term V_1 has the same form as in Fridman and Shaked (2001) and Fridman (2001) and is actually

$$V_1 = x^{\mathrm{T}}(t)P_1x(t).$$

For the proof of the theorem, it is sufficient to show that $J_{zw} < 0$ for any nonzero w and $\dot{V} < 0$ for $w = 0$. Using the descriptor system approach introduced in Fridman and Shaked (2002), the time derivative of V_1 along the solution of (5) is given by

$$\dot{V}_1 = 2\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^{\mathrm{T}} P^{\mathrm{T}} \begin{bmatrix} \dot{x}(t) \\ 0 \end{bmatrix}$$
$$= 2\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^{\mathrm{T}} P^{\mathrm{T}} \begin{bmatrix} \dot{x}(t) \\ -\dot{x}(t) + Ax(t) + A_1x(t-h) + B_ww(t) \end{bmatrix}.$$

$\dot{V}_4$ is represented as follows:

$$\dot{V}_4 = h\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} X_{11} & X_{12} \\ \star & X_{22} \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} + 2\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \times [x(t) - x(t-h)] + \int_{t-h}^{t} \dot{x}(\alpha) Z \dot{x}(\alpha)\,\mathrm{d}\alpha.$$

We can rewrite J_{zw} as follows:

$$J_{zw} = \int_0^\infty [z^{\mathrm{T}}(t)z(t) - \gamma^2 w^{\mathrm{T}}(t)w(t) + \dot{V}(x_t)]\,\mathrm{d}t + V(x_t)|_{t=0} - V(x_t)|_{t=\infty}.$$

Because $V(x_t)|_{t=0} = 0$ under zero initial condition and $V(x_t)|_{t=\infty} \geqslant 0$, we lead to

$$J_{zw} \leqslant \int_0^\infty [z^{\mathrm{T}}(t)z(t) - \gamma^2 w^{\mathrm{T}}(t)w(t) + \dot{V}(x_t)]\,\mathrm{d}t.$$

Using the relation $h \leqslant \bar{h}$, the upper bound on J_{zw} is written as follows:

$$J_{zw} \leqslant \int_0^\infty [z^{\mathrm{T}}(t)z(t) - \gamma^2 w^{\mathrm{T}}w(t) + \dot{V}(x_t)]\,\mathrm{d}t$$

$$\leqslant \int_0^\infty \begin{bmatrix} x(t) \\ \dot{x}(t) \\ x(t-h) \\ w(t) \end{bmatrix}^{\mathrm{T}} \times \underbrace{\begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} \\ \star & -Q + C_1^{\mathrm{T}}C_1 & 0 \\ \star & \star & -\gamma^2 I + D_w^{\mathrm{T}} D_w \end{bmatrix}}_{\Xi} \times \begin{bmatrix} x(t) \\ \dot{x}(t) \\ x(t-h) \\ w(t) \end{bmatrix} \mathrm{d}t,$$

where

$$\Psi_{11} = \begin{bmatrix} 0 & I \\ A & -I \end{bmatrix}^{\mathrm{T}} P + P^{\mathrm{T}} \begin{bmatrix} 0 & I \\ A & -I \end{bmatrix} + \bar{h}\begin{bmatrix} X_{11} & X_{12} \\ \star & X_{22} \end{bmatrix} + \begin{bmatrix} Q + C^{\mathrm{T}}C + Y_1 + Y_1^{\mathrm{T}} & Y_2^{\mathrm{T}} \\ \star & \bar{h}Z \end{bmatrix}$$

$$\Psi_{12} = P^{\mathrm{T}}\begin{bmatrix} 0 \\ A_1 \end{bmatrix} - \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}, \quad \Psi_{13} = P^{\mathrm{T}}\begin{bmatrix} 0 \\ B_w \end{bmatrix} + \begin{bmatrix} C^{\mathrm{T}}D_w \\ 0 \end{bmatrix}.$$

$\Xi < 0$ implies that $J_{zw} < 0$. After some manipulation using Schur complement, the inequality $\Xi < 0$ is equivalently changed to (6). Consider the following inequality:

$$\begin{bmatrix} \Gamma_{11} & \Gamma_{12} & P_2^{\mathrm{T}}A_1 - Y_1 \\ \star & \Gamma_{22} & P_3^{\mathrm{T}}A_1 - Y_2 \\ \star & \star & -Q \end{bmatrix} < 0, \tag{9}$$

which guarantees $\dot{V} < 0$ in case of $w = 0$. If the LMI (6) is feasible, then the LMI (9) is also feasible. Thus, the system (5) is stable. This completes the proof. □

Remark 3.1. The conditions (6) and (7) are LMI conditions. Therefore, for fixed γ, the maximum upper bound of the allowable delay $\bar{h}$ can be obtained efficiently using existing convex optimization algorithms. Similarly, for fixed $\bar{h}$, we can find the minimum value of γ.

Remark 3.2. Some sort of model transformation and bounding technique for cross terms are commonly used in the derivation of existing delay-dependent results. It is noted that those are unnecessary in the proposed approach and hence the derivation procedure is much simpler.

4. H_∞ control for nominal systems

In this section, we develop delay-dependent H_∞ control for nominal systems based on the bounded real lemma derived in the previous section. Consider nominal systems with a state-delay

$$\dot{x}(t) = Ax(t) + A_1 x(t-h) + Bu(t) + B_w w(t),$$

$$z(t) = \begin{bmatrix} Cx(t) + D_w w(t) \\ C_1 x(t-h) \\ Du(t) \end{bmatrix},$$

$$x(t) = 0, \quad t \in [-\bar{h}, 0]. \tag{10}$$

We present a sufficient condition under which there exists a memoryless state-feedback H_∞ controller for nominal systems (10).

Theorem 4.1. *For some prescribed $\gamma > 0$ and $\bar{h} \geqslant 0$, assume that there exist $L_1 > 0$, L_2, L_3, W, M_{11}, M_{12}, M_{22}, N_1, N_2, R, and V such that*

$$\begin{bmatrix} \Pi_{11} & \Pi_{12} & -N_1 & 0 & L_1C^{\mathrm{T}} & 0 & V^{\mathrm{T}}D^{\mathrm{T}} & \bar{h}L_2^{\mathrm{T}} \\ \star & \Pi_{22} & A_1L_1 - N_2 & B_w & 0 & 0 & 0 & \bar{h}L_3^{\mathrm{T}} \\ \star & \star & -W & 0 & 0 & L_1C_1^{\mathrm{T}} & 0 & 0 \\ \star & \star & \star & -\gamma^2 I & D_w^{\mathrm{T}} & 0 & 0 & 0 \\ \star & \star & \star & \star & -I & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -I & 0 & 0 \\ \star & \star & \star & \star & \star & \star & -I & 0 \\ \star & \star & \star & \star & \star & \star & \star & -\bar{h}R \end{bmatrix} < 0, \tag{11}$$

$$\begin{bmatrix} M_{11} & M_{12} & N_1 \\ \star & M_{22} & N_2 \\ \star & \star & L_1R^{-1}L_1 \end{bmatrix} \geqslant 0, \tag{12}$$

where

$$\Pi_{11} \triangleq L_2 + L_2^{\mathrm{T}} + W + N_1 + N_1^{\mathrm{T}} + \bar{h}M_{11},$$

$$\Pi_{12} \triangleq (AL_1 + BV)^{\mathrm{T}} - L_2^{\mathrm{T}} + L_3 + N_2^{\mathrm{T}} + \bar{h}M_{12},$$

$$\Pi_{22} \triangleq -L_3 - L_3^{\mathrm{T}} + \bar{h}M_{22}.$$

Then system (10) *controlled by* $u(t) = VL_1^{-1}x(t)$ *is stable and satisfies* $\|z\|_2 < \gamma\|w\|_2$ *for all nonzero* $w \in L_2 \in [0,\infty)$ *and any constant time-delay* h *satisfying* $0 \leqslant h \leqslant \bar{h}$.

Proof. Assuming $u(t) = Kx(t)$, A and $Du(t)$ in (10) are replaced by $(A+BK)$ and $DKx(t)$, respectively. Taking this into account, the condition $\Xi < 0$ is replaced by

$$\begin{bmatrix} \hat{\Psi}_{11} & P^{\mathrm{T}}\begin{bmatrix} 0 \\ A_1 \end{bmatrix} - \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} & P^{\mathrm{T}}\begin{bmatrix} 0 \\ B_w \end{bmatrix} + \begin{bmatrix} C^{\mathrm{T}}D_w \\ 0 \end{bmatrix} \\ \star & -Q + C_1^{\mathrm{T}}C_1 & 0 \\ \star & \star & -\gamma^2 I + D_w^{\mathrm{T}}D_w \end{bmatrix} < 0, \tag{13}$$

where

$$\hat{\Psi}_{11} = \begin{bmatrix} 0 & I \\ (A+BK) & -I \end{bmatrix}^{\mathrm{T}} P + P^{\mathrm{T}}\begin{bmatrix} 0 & I \\ (A+BK) & -I \end{bmatrix} + \bar{h}\begin{bmatrix} X_{11} & X_{12} \\ \star & X_{22} \end{bmatrix} + \begin{bmatrix} Q + C^{\mathrm{T}}C + K^{\mathrm{T}}D^{\mathrm{T}}DK + Y_1 + Y_1^{\mathrm{T}} & Y_2^{\mathrm{T}} \\ \star & \bar{h}Z \end{bmatrix}.$$

Using the idea in Fridman and Shaked (2002), define

$$P^{-1} = L = \begin{bmatrix} L_1 & 0 \\ L_2 & L_3 \end{bmatrix}.$$

Pre- and post-multiply (13) by $\mathrm{diag}\{L^{\mathrm{T}}, L_1, I\}$ and by $\mathrm{diag}\{L, L_1, I\}$, respectively. Similarly, pre- and post-multiply (7) by $\mathrm{diag}\{L^{\mathrm{T}}, L_1\}$ and $\mathrm{diag}\{L, L_1\}$, respectively and introduce change of variables such that

$$\begin{bmatrix} M_{11} & M_{12} \\ \star & M_{22} \end{bmatrix} \triangleq L^{\mathrm{T}}\begin{bmatrix} X_{11} & X_{12} \\ \star & X_{22} \end{bmatrix} L,$$

$$\begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \triangleq L^{\mathrm{T}}\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} L_1,$$

$$W \triangleq L_1QL_1, \quad R \triangleq Z^{-1}, \quad V = KL_1.$$

After some manipulation including the Schur complement, we can represent conditions (13) and (7) by (11) and(12), respectively. This completes the proof. □

It is noted that the conditions in Theorem 4.1 are no longer LMI conditions because of the term $L_1R^{-1}L_1$ in (12). As a result, unfortunately in this case, we cannot find a maximum $\bar{h}$ or minimum γ using convex optimization algorithms. A suboptimal maximum delay $\bar{h}$ or minimum γ can be simply obtained by setting $R = L_1$ in (11) and (12), which results in LMI conditions. However, with more computational effort, better results can be obtained using an iterative algorithm, which is presented next.

The form of nonlinearity $L_1R^{-1}L_1$ in (12) coincides with the one introduced in Moon et al. (2001). Therefore, following the similar step taken in Moon et al. (2001), we can convert the original nonconvex feasibility problem characterized by the matrix inequalities (11) and (12) to the following nonlinear minimization problem subject to LMIs:

Minimize $\mathrm{tr}(ST + L_1J + RG)$

subject to (11) and (14)

$$L_1 > 0, \quad S > 0, \quad \begin{bmatrix} M_{11} & M_{12} & N_1 \\ \star & M_{22} & N_2 \\ \star & \star & S \end{bmatrix} \geqslant 0,$$

$$\begin{bmatrix} T & J \\ J & G \end{bmatrix} > 0, \tag{15}$$

$$\begin{bmatrix} S & I \\ I & T \end{bmatrix} \geqslant 0, \quad \begin{bmatrix} L_1 & I \\ I & J \end{bmatrix} \geqslant 0, \quad \begin{bmatrix} R & I \\ I & G \end{bmatrix} \geqslant 0. \tag{16}$$

If the solution of the above minimization problem is $3n$, that is, $\mathrm{tr}(ST + L_1J + RG) = 3n$, we can say from Theorem 4.1 that the closed-loop system (10) controlled by $u(t) = VL_1^{-1}x(t)$ satisfies $\|z(t)\|_2 < \gamma\|w(t)\|_2$. Actually, utilizing the linearization method (El Ghaoui, Oustry, & Ait Rami, 1997), we can find a suboptimal maximum delay for some prescribed γ or can find a suboptimal minimum γ for some prescribed $\bar{h}$ relatively easily using an iterative algorithm presented below. Note that condition (12) is used as a stopping criterion in the algorithm. This is because our purpose is to find a feasible solution satisfying inequalities (11) and (12), not to find a solution such that $\mathrm{tr}(ST + L_1J + RG)$ is exactly equal to $3n$.

Algorithm.

(1) Choose a sufficiently small initial $\bar{h} > 0$ (Choose a sufficiently large initial $\gamma > 0$) such that there exists a feasible solution to (14), (15), and (16). Set $\bar{h}_{\mathrm{so}} = \bar{h}$ (Set $\gamma_{\mathrm{so}} = \gamma$).

(2) Find a feasible set $(L_1^0, L_2^0, L_3^0, M_{11}^0, M_{12}^0, M_{22}^0, N_1^0, N_2^0, W^0, R^0, V^0, S^0, T^0, J^0, G^0)$ satisfying (14), (15), and (16). Set $k = 0$.

(3) Solve the following LMI problem for the variables $(L_1, L_2, L_3, M_{11}, M_{12}, M_{22}, N_1, N_2, W, R, V, S, T, J, G)$

Minimize $\mathrm{tr}(S^k T + T^k S + L_1^k J + J^k L_1 + R^k G + G^k R)$

subject to (14), (15), and (16).

Set $S^{k+1} = S$, $T^{k+1} = T$, $L_1^{k+1} = L_1$, $J^{k+1} = J$, $R^{k+1} = R$, and $G^{k+1} = G$.

(4) If condition (12) is satisfied, then set $\bar{h}_{so} = \bar{h}$ (set $\gamma_{so} = \gamma$) and return to Step 2 after increasing $\bar{h}$ (decreasing γ) to some extent. If condition (12) is not satisfied within a specified number of iterations, say $k_{\max}$, then exit. Otherwise, set $k = k + 1$ and go to Step 3.

Using the above algorithm, we can obtain a suboptimal maximum of $\bar{h}$ (a suboptimal minimum of γ). Later in Section 6 we illustrate, using numerical examples, that the above algorithm can provide satisfactory results. In the following section, we extend the H_∞ control obtained for nominal systems to that for uncertain systems (1).

5. Robust H_∞ control for uncertain systems

In this section, we consider the robust H_∞ control for uncertain systems (1). For that purpose, we extend Theorem 4.1 to that for uncertain systems (1).

Theorem 5.1. *For some prescribed $\gamma > 0$ and $\bar{h} \geqslant 0$, assume that there exist $L_1 > 0$, L_2, L_3, M_{11}, M_{12}, M_{22}, N_1, N_2, W, R, V, and $\Lambda = \mathrm{diag}\{\lambda_1 I, \ldots, \lambda_r I\}$ such that*

$$\begin{bmatrix} \Omega_{11} & \Omega_{12} & -N_1 & 0 & L_1C^{\mathrm{T}} & 0 & V^{\mathrm{T}}D^{\mathrm{T}} & \bar{h}L_2^{\mathrm{T}} & (EL_1 + E_bV)^{\mathrm{T}} \\ \star & \Omega_{22} & A_1L_1 - N_2 & B_w & 0 & 0 & 0 & \bar{h}L_3^{\mathrm{T}} & 0 \\ \star & \star & -W & 0 & 0 & L_1C_1^{\mathrm{T}} & 0 & 0 & L_1E_1^{\mathrm{T}} \\ \star & \star & \star & -\gamma^2 I & D_w^{\mathrm{T}} & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & -I & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -I & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & -I & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & -\bar{h}R & 0 \\ \star & \star & \star & \star & \star & \star & \star & \star & -\Lambda \end{bmatrix} < 0, \qquad (17)$$

$$\begin{bmatrix} M_{11} & M_{12} & N_1 \\ \star & M_{22} & N_2 \\ \star & \star & L_1R^{-1}L_1 \end{bmatrix} \geqslant 0, \qquad (18)$$

where

$$\Omega_{11} \triangleq L_2 + L_2^{\mathrm{T}} + W + N_1 + N_1^{\mathrm{T}} + \bar{h}M_{11},$$

$$\Omega_{12} \triangleq (AL_1 + BV)^{\mathrm{T}} - L_2^{\mathrm{T}} + L_3 + N_2^{\mathrm{T}} + \bar{h}M_{12},$$

$$\Omega_{22} \triangleq -L_3 - L_3^{\mathrm{T}} + \bar{h}M_{22} + F\Lambda F^{\mathrm{T}}.$$

Then the uncertain system (1) controlled by $u(t) = VL_1^{-1}x(t)$ is stable and satisfies $\|z\|_2 < \gamma\|w\|_2$ for all nonzero $w \in L_2[0, \infty)$ and any constant time-delay h satisfying $0 \leqslant h \leqslant \bar{h}$.

Proof. It is sufficient for the proof of Theorem 5.1 to show that (11) is still satisfied even with A, A_1, and B replaced by $A + F\Delta E$, $A_1 + F\Delta E_1$ and $B + F\Delta E_b$, respectively. Define the matrix in left side of '<' in (11) to be Φ. Then, condition (11) with A, A_1, and B replaced by $A + F\Delta E$, $A_1 + F\Delta E_1$ and $B + F\Delta E_b$, respectively, is written as

$$\Phi + \bar{F}\Delta\bar{E} + \bar{E}^{\mathrm{T}}\Delta^{\mathrm{T}}\bar{F}^{\mathrm{T}} < 0, \qquad (19)$$

where

$$\bar{F} \triangleq [0 \;\; F^{\mathrm{T}} \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0]^{\mathrm{T}},$$

$$\bar{E} \triangleq [EL_1 + E_bV \;\; 0 \;\; E_1L_1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0].$$

According to (3) in Lemma 2.1, (19) holds if there exists

$$\Lambda = \mathrm{diag}\{\lambda_1 I, \ldots, \lambda_r I\} > 0.$$

such that

$$\Phi + \bar{F}\Lambda\bar{F}^{\mathrm{T}} + \bar{E}^{\mathrm{T}}\Lambda^{-1}\bar{E} < 0. \qquad (20)$$

By Schur complement, (20) is equivalent to (17). This completes the proof. □

For Theorem 5.1, we can construct an iterative algorithm in a similar way given in the previous section. In the subsequent section, numerical examples will be given to illustrate the less conservatism of the proposed methods.

6. Numerical examples

Consider the linear uncertain time-delay system taken from de Souza and Li (1999):

$$\begin{cases} \dot{x}(t) = [A + \Delta A(t)]x(t) + [A_1 + \Delta A_1(t)]x(t-h) \\ \qquad + Bu(t) + B_w w(t), \\ z(t) = \begin{bmatrix} Cx(t) \\ Du(t) \end{bmatrix}, \end{cases} \qquad (21)$$

where

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & -1 \\ 0 & -0.9 \end{bmatrix},$$

$$B_w = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [0 \;\; 1].$$

In Fridman and Shaked (2002), the case where $\Delta A(t) = \Delta A_1(t) = 0$ and $D = 0.1$ was considered and it is reported that, for $\bar{h} = 0.999$, a minimum of $\gamma = 0.1287$ with a corresponding gain $K = [0 \;\; -1.0285 \times 10^6]$ was obtained.

For the same system, we applied Theorem 4.1. Tables 1 and 2 show the controller gains of the H_∞ controllers obtained and the number of iterations for some prescribed value of $\bar{h}$ and γ, respectively. The number of iterations denotes the iterations counted until the stopping criterion, i.e. condition (12), was activated. Table 1 shows that, for $\bar{h} = 0.999$, the proposed method can produce an H_∞ controller even for $\gamma = 0.1015$, which is much smaller than the value given in Fridman and Shaked (2002). It is also an important observation that the controller gains are much smaller than that given in Fridman and Shaked (2002). Because the high control gain is very likely to lead to the input magnitude saturation, smaller gain for the same γ seems desirable. Table 2 shows that, for $\gamma = 0.1287$, the system can be stabilized for any time delay $h \leqslant 1.25$.

In de Souza and Li (1999), $\Delta A(t)$ and $\Delta A_1(t)$ are uncertain matrices satisfying $\|\Delta A(t)\| \leqslant 0.2$, $\|\Delta A_1(t)\| \leqslant 0.2$, $\forall t \geqslant 0$ and $D = 0$. In this case, the system (21) is represented by (1) with

$$F = \begin{bmatrix} 0.2 & 0 & 0.2 & 0 \\ 0 & 0.2 & 0 & 0.2 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}^{\mathrm{T}},$$

$$E_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}},$$

$$E_b = [0 \;\; 0 \;\; 0 \;\; 0]^{\mathrm{T}}, \quad \Delta(t) = \begin{bmatrix} \Delta_1(t) & 0 \\ 0 & \Delta_2(t) \end{bmatrix},$$

where $\Delta_1(t), \Delta_2(t) \in \mathbf{R}^{2\times 2}$. It is reported that, applying the method in de Souza and Li (1999), system (21) is robustly stabilizable for $\bar{h} = 0.3346$. When $\bar{h} = 0.3$, the smallest γ obtainable was 1.95. When $\bar{h}$ was decreased to 0.2, the smallest obtainable γ decreased to 0.66.

We applied Theorem 5.1 to this uncertain system. For $\bar{h} = 0.8$ and $\gamma = 0.05$, the proposed method yielded the controller

$$u(t) = [-0.0337 - 64.9821]\, x(t)$$

after 29 iterations. Note that the proposed method provides H_∞ controller achieving much smaller γ for much bigger $\bar{h}$, which implies that the proposed method is much less conservative than the existing results.

Table 1
H_∞ controller of Theorem 4.1 for $\bar{h} = 0.999$

γ	Feedback gain	Number of iterations
0.1287	[− 0.0054 − 29.2947]	15
0.105	[0.9885 − 142.7580]	21
0.102	[2.5712 − 511.5304]	52
0.1015	[3.6828 − 827.0898]	74

Table 2
H_∞ controller of Theorem 4.1 for $\gamma = 0.1287$

$\bar{h}$	Feedback gain	Number of iterations
1.1	[0.1686 − 36.5849]	19
1.2	[0.4514 − 55.9983]	32
1.25	[0.6407 − 89.1149]	86

7. Conclusions

This paper proposed a new delay-dependent robust H_∞ control for uncertain systems with a state-delay. First, we presented a new delay-dependent bounded real lemma for systems with a state-delay. For that purpose, we chose a new Lyapunov–Krasovskii functional, with which we need neither model transformation nor bounding for cross terms in deriving delay-dependent results. Based on the bounded real lemma obtained, the condition for the existence of H_∞ control was given in terms of matrix inequalities including nonlinear terms, which results in a nonconvex feasibility problem. Because the original nonconvex feasibility problem is hard to solve, we provided an iterative algorithm using convex optimization as an alternative solution. Although somewhat big computational burden is required to obtain a controller, the proposed results are much less conservative than the existing results in that the obtained controller allows a larger delay bound for a fixed performance bound and achieves a lower performance bound for a fixed time-delay. This decreased conservatism was clearly demonstrated using numerical examples.

References

Choi, H. H., & Chung, M. J. (1997). An LMI approach to H_∞ controller design for linear time-delay systems. *Automatica*, *33*(4), 737–739.

de Souza, C. E., & Li, X. (1999). Delay-dependent robust H_∞ control of uncertain linear state-delayed systems. *Automatica*, *35*, 1313–1321.

El Ghaoui, L., Oustry, F., & Ait Rami, M. (1997). A cone complementarity linearization algorithm for static output-feedback and related problems. *IEEE Transactions on Automatic Control*, *42*(8), 1171–1176.

Fridman, E. (2001). New Lyapunov-Krasovskii functional for stability of linear retarded and neutral type systems. *Systems & Control Letters*, *43*, 309–319.

Fridman, E., & Shaked, U. (2001). New bounded real lemma representations for time-delay systems and their applications. *IEEE Transactions on Automatic Control*, *46*, 1973.

Fridman, E., & Shaked, U. (2002). A descriptor system approach to H_∞ control of linear time-delay systems. *IEEE Transactions on Automatic Control*, *47*(2), 253–270.

Kapila, V., & Haddad, W. M. (1998). Memoryless H_∞ controllers for discrete-time systems with time delay. *Automatica*, *34*(9), 1141–1144.

Kim, J. H., & Park, H. B. (1999). H_∞ state feedback control for generalized continuous/discrete time-delay systems. *Automatica*, *35*, 1443–1451.

Lee, Y. S., Moon, Y. S., & Kwon, W. H. (2001). Delay-dependent robust H_∞ control for uncertain systems with time-varying state-delay. In: *Proceedings of conference on decision and control*, San Diego, CA (pp. 3208–3213).

Moon, Y. S. (1998). *Robust control of time-delay systems using linear matrix inequalities*. Ph.D. thesis, School of Electrical Engineering and Computer Sciences, Seoul National Univ.

Moon, Y. S., Park, P. G., Kwon, W. H., & Lee, Y. S. (2001). Delay-dependent robust stabilization of uncertain state-delayed systems. *International Journal of Control*, *74*(14), 1447–1455.

Park, P. (1999). A delay-dependent stability criterion for systems with uncertain time-invariant delays. *IEEE Transactions on Automatic Control*.

Park, P. G., Moon, Y. S., & Kwon, W. H. (1998). A delay-dependent robust stability criterion for uncertain time-delay systems. In *Proceedings of the American control conference* (pp. 1963–1964).

Zribi, M., & Mahmoud, M. S. (1999). H_∞-control design for systems with multiple delays. *Computers and Electrical Engineering*, *25*, 451–475.

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