A Constrained Receding Horizon Estimator with FIR Structures

Pyung Soo Kim and Young Sam Lee

Abstract: This paper concerns with a receding horizon estimator (RHE) for discrete-time linear systems subject to *constraints* on the estimate. In solving the optimization for every horizons, the past all measurement data outside the horizon is discarded and thus the arrival cost is not considered. The RHE in the current work is a finite impulse response (FIR) structure which has some good inherent properties. The proposed RHE can be represented in the simple matrix form for the unconstrained case. Various numerical examples demonstrate how including constraints in the RHE can improve estimation performance. Especially, in the application to the unknown input estimation, it will be shown how the FIR structure in the RHE can improve the estimation speed.

Keywords: state estimation, constraints, FIR structure, quadratic programming

I. Introduction

The Kalman filter has been an important tool for the last thirty years, not only for control system design, but also for many other fields of engineering and applied science [1]. Often additional insight about the system is available in the form of inequality constraints. However, with the addition of the inequality constraints, general recursive solutions such as the Kalman filter are unavailable.

Although there exists a vase literature addressing estimation, relatively little work has been carried out for systems in which the estimated variables must satisfy *a priori* constraints. If the data are processed in batch fashion, inequality constraints can be easily be incorporated within least squares estimation using a quadratic programming. However, the problem size grows with time as more data becomes available. Thus, its on-line application might be limited. To make the estimation problem tractable, a receding or a moving horizon formulation has been proposed where the least squares optimization is performed over a fixed length horizon to bound the size of the quadratic program [2] [3]. The obtained estimator will be called the receding horizon estimator (RHE). In the existing RHE [2] [3], to summarize compactly the effect of the past all measurement data outside the horizon, the arrival cost has been considered in the optimization. That is, the past data outside the horizon affects the optimization in the current horizon. Therefore, although this RHE can incorporate inequality constraints, it still has an infinite impulse response (IIR) structure. It has been a general rule of thumb that the IIR structure filter such as the Kalman filter is often sensitive against temporarily uncertain modeling errors or numerical errors [4]-[6].

In this paper, an alternative RHE will be investigated for discrete-time linear systems with inequality constraints. This estimator will have a finite impulse response (FIR) structure which utilizes only the finite measurement data on the most recent horizon. The FIR structure in filters has been adopted due to its inherent properties such as a bounded input/bounded output (BIBO) stability, robustness for temporary modeling uncertainties and numerical errors [4]-[6]. To be an FIR structure, in solving the optimization, only the finite measurement data on the horizon is utilized while the past all measurement data outside the horizon is discarded. That is, the arrival cost term is not considered by taking the covariance matrix of the horizon initial state as infinity. This means that the horizon initial state is assumed to be unknown when solving the optimization for every horizons. It is shown that the proposed RHE can be represented in the simple matrix form for the unconstrained case. Various numerical examples demonstrate how including constraints in the RHE can improve estimation performance. Especially, in the application to the unknown input estimation on a continuous stirred tank reactor (CSTR) model, it will be shown how the FIR structure in the RHE can improve the tracking speed.

II. Problem statements

Consider a linear discrete time-invariant state-space model:

$$x_{k+1} = Ax_k + Gw_k, \quad y_k = Cx_k + v_k \tag{1}$$

where $x_k \in \Re^n$ is the state vector and $y_k \in \Re^q$ is the measured output vector, respectively. The initial state x_{k_0} is a random variable with a mean $\bar{x}_{k_0}$ and a covariance Π_{k_0}. The noise terms $w_k \in \Re^p$ and $v_k \in \Re^q$ are random variables with zero mean and covariances Q and R respectively, and mutually uncorrelated. The random variable w_k typically models unmeasured disturbances and model inaccuracies, while the random variable v_k is measurement noise. There two variables are uncorrelated with x_{k_0}.

Although forming an accurate probabilistic model for the unknown variables such as states x_k and disturbances w_k is difficult, an engineer will usually have knowledge about the range of values that they can assume. This knowledge can be formed as inequality consraints as follows:

$$\begin{aligned} w_k \in \mathcal{W} &\triangleq \{w : Dw \le d\}, \\ x_k \in \mathcal{X} &\triangleq \{x : Hx \le h\} \end{aligned} \tag{2}$$

where $D \in \Re^{n_w \times n}$, $H \in \Re^{n_x \times n}$, $d \in \Re^{n_w}$ and $h \in \Re^{n_x}$. The capability to incorporate above constraints on the estimated variables have lead to better estimates [2] [3]. When optimization software such as quadratic programming or nonlinear programming is used to solve the least squares problem, inequality

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constraints (2) can be placed on the unknown variables. This is useful form an engineering viewpoint since the prior knowledge of the process is often in the form of inequalities. For instance. variables such as temperature, pressure, flow rates, concentrations, etc. must be nonnegative and cannot go above some upper bound. In addition, the rate-of-change of these variables is also bounded by mass and energy balance considerations.

Although the quadratic programming can incorporate inequality constraints of (2), its on-line application is limited since the size of the problem grows as more data become available. Thus, for a fixed dimension quadratic program, a receding or moving horizon formulation can be a strategy [2] [3]. That is, for the current time k, the optimization problem on the interval $[k_0, k]$ with varied horizon length $k - k_0$ is reformulated on the most recent horizon $[k-N, k]$ with fixed horizon length N. For compactness, $k_N \triangleq k - N$ shall be written hereafter. In the current paper, the estimator with a receding or moving horizon strategy will be called the receding horizon estimator (RHE). However, in the existing RHE [2] [3], due to the arrival cost term which compactly summarizes the effect of the past all measurement data outside the horizon, the RHE is an infinite impulse response (IIR) structure which utilizes all of the available measurement data. It has been a general rule of thumb that the IIR structure filter such as the Kalman filter is often sensitive against temporarily uncertain modeling errors or numerical errors [4]-[6].

III. Constrained RHE with FIR structure

In this section, the RHE with an FIR structure will be investigated subject to constraints on the estimate. To be an FIR structure, in solving the optimization on the current horizon $[k_N, k]$, only the finite measurement data on the horizon is utilized while the past all measurement data outside the horizon is discarded. This means that the arrival cost term in [2] [3] will not be considered in optimization to obtain the optimal state and disturbance estimates.

On the most recent horizon$[k_N, k]$, the system (1) will be represented in a batch form that the finite measurement data is expressed in terms of the state x_k at the current time k as follows:

$$\begin{aligned} Y_{k-1} &= \bar{C}_N x_k + \bar{G}_N W_{k-1} + V_{k-1} \\ &= \begin{bmatrix} \bar{C}_N & \bar{G}_N \end{bmatrix} \begin{bmatrix} x_k \\ W_{k-1} \end{bmatrix} + V_{k-1} \end{aligned} \tag{3}$$

where $Y_{k-1} \triangleq [y_{k_N}^T \cdots y_{k-1}^T]^T$, $W_{k-1} \triangleq [w_{k_N}^T \cdots w_{k-1}^T]^T$, $V_{k-1} \triangleq [v_{k_N}^T \cdots v_{k-1}^T]^T$ and $\bar{C}_N$, $\bar{G}_N$ are obtained from

$$\bar{C}_N \triangleq \begin{bmatrix} CA^{-N} \\ CA^{-N+1} \\ \vdots \\ CA^{-1} \end{bmatrix},$$

$$\bar{G} \triangleq -\begin{bmatrix} CA^{-1}G & CA^{-2}G & \cdots & CA^{-N}G \\ 0 & CA^{-1}G & \cdots & CA^{-N+1}G \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & CA^{-1}G \end{bmatrix}.$$

On the horizon $[k_N, k]$ for the current time k, the receding horizon estimator (RHE) is obtained from the solution of the following quadratic program

$$J_N^* = \min_{x_{k|k-1}, W_{k-1}} J_N(x_{k|k-1}, W_{k-1})$$

subject to (1) and inequality constraints (2) where

$$\begin{aligned} & J_N(x_{k|k-1}, W_{k-1}) \\ = & \begin{bmatrix} Y_{k-1} - \begin{bmatrix} \bar{C}_N & \bar{G}_N \end{bmatrix} \begin{bmatrix} x_{k|k-1} \\ W_{k-1} \end{bmatrix} \\ W_{k-1} \end{bmatrix}^T \\ & \begin{bmatrix} \bar{R}_N & 0 \\ 0 & \bar{Q}_N \end{bmatrix}^{-1} \\ & \begin{bmatrix} Y_{k-1} - \begin{bmatrix} \bar{C}_N & \bar{G}_N \end{bmatrix} \begin{bmatrix} x_{k|k-1} \\ W_{k-1} \end{bmatrix} \\ W_{k-1} \end{bmatrix} \end{aligned} \tag{4}$$

with weighting matrices given by $\bar{R}_N \triangleq [\mathrm{diag}(Q \cdots Q)]$ and $\bar{Q}_N \triangleq [\mathrm{diag}(R \cdots R)]$. Then, on the most recent horizon $[k_N, k]$, the optimal state and disturbance estimates are denoted at time k by $\hat{x}_{k|k-1}$ and $\hat{W}_{k-1}$ given measurement data Y_{k-1}. In particular, if

$$(\hat{x}_{k|k-1}, \hat{W}_{k-1}) \in arg \min_{x_{k_N}, W_{k-1}} J_N(x_{k_N}, W_{k-1}),$$

then the RHE $\hat{x}_{k|k-1}$ denotes the solution to (4) at time k.

The obtained RHE $\hat{x}_{k|k-1}$ has the FIR structure since only the finite measurement data Y_{k-1} on the most recent horizon $[k_N, k]$ is utilized. The FIR structure in filters has known to be built in a bounded input/bounded output (BIBO) stability and to be robust against temporarily uncertain model parameters [4]-[6]. Therefore, the RHE with FIR structure might have above inherent properties of the FIR structure filter. In the system for detecting a signal with unknown time of occurrence, it is well known that increasing of measurements for a detection decision will increase the time to detection, i.e., an increase in a delay from the time unknown signal first appears would be appropriate for quick detection of signals with unknown times of occurrence, which will be shown via a real application in the following section.

Note that the proposed RHE with FIR structure can be represented in a simple matrix form when there are no constraints as follows:

$$\begin{aligned} \hat{x}_{k|k-1} &= \begin{bmatrix} \bar{C}_N^T \bar{R}_N^{-1} \bar{C}_N & \bar{C}_N^T \bar{R}_N^{-1} \bar{G}_N \\ \bar{G}_N^T \bar{R}_N^{-1} \bar{C}_N & \bar{G}_N^T \bar{R}_N^{-1} \bar{G}_N + \bar{Q}_N^{-1} \end{bmatrix}^{-1} \\ & \bar{C}_N^T \bar{R}_N^{-1} Y_{k-1}. \end{aligned} \tag{5}$$

which is obtained easily from the the minimization of (4).

IV. Numerical examples

1. Simple example of inequality constraints

To illustrate the validity of the constrained estimation, simple numerical example is implemented on a following discrete-time linear system:

$$\begin{aligned} x_{k+1} &= \begin{bmatrix} 0.9962 & 0.1949 \\ -0.1949 & 0.3815 \end{bmatrix} x_k + \begin{bmatrix} 0.03393 \\ 0.1949 \end{bmatrix} w_k, \\ y_k &= \begin{bmatrix} 1 & -3 \end{bmatrix} x_k + v_k \end{aligned}$$

where v_k is zero mean and normally distributed random variable with covariance 0.01^2, and $w_k = |z_k|$ where z_k is zero

mean and normally distributed random variable with covariance 0.1^2. The initial state x_0 is normally distributed with zero mean and covariance equal to the identity. The constrained estimation problem is formulated with $Q = 0.1^2$, $R = 0.01^2$ and $N = 10$. To capture our knowledge of the random variable w_k, the inequality constraints $w_k \geq 0$ is enforced. Four estimators are compared. The first is unconstrained RHE with IIR structure, which is the Kalman filter [3]. The second is the constrained RHE with IIR structure [3]. The third is the unconstrained RHE with FIR structure, which is given by (5). Finally, fourth is the constrained RHE with FIR structure, which is the main result of this paper. The result is shown in Fig. 1. As expected, the performance of two constrained RHEs are superior to other two unconstrained RHEs, since they possess more information regarding the random variable in the form of equality constraints. It is remarkable that the constrained RHE with FIR structure shows similar performance to the constrained RHE with IIR structure, although the arrival cost is not considered in optimization.

2. Application to unknown input estimation

To show the useful application of the constrained RHE with FIR structure, the problem of an unknown input estimation is considered. The unknown input estimation has been applied to many engineering problems [7] [8]. It has been shown that the unknown input estimation using the FIR structure filter can provide quicker estimation than the approach using IIR filter such as the Kalman filter [8]. In this section, the application of the unknown input estimation demonstrates how including constraints in the RHE can improve estimation performance. In addition, it will be shown how the FIR structure in the RHE can improve the tracking speed.

The unknown input estimation using the constrained RHE with FIR structure is performed for a continuous stirred tank reactor (CSTR). The unknown input vector with a random-walk type is treated as auxiliary state. Then, the CSTR system can be augmented as the following fourth order system:

$$\begin{aligned} x_{k+1} &= \begin{bmatrix} 0.9534 & 3.5868 & -2.4413 & 0.0449 \\ 0.0015 & 0.6152 & 0.0354 & 0.0000 \\ 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 1.0000 \end{bmatrix} x_k \\ &\quad + \begin{bmatrix} -0.0622 & 0 & 0 \\ 0.0010 & 0 & 0 \\ 0 & 0.0500 & \\ 0 & 0 & 0.0500 \end{bmatrix} w_k, \\ y_k &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} x_k + v_k \end{aligned} \tag{6}$$

with inequality constraints which should be enforced include:

$$\begin{aligned} &-x_2 \leq 0.051, \;\; x_2 \leq (1 - 0.051), \\ &x_3 \leq 0, \;\; x_4 \leq 0, \;\; -x_3 \leq 0.5. \end{aligned} \tag{7}$$

In the augmented state $x_k = [x_1 \; x_2 \; x_3 \; x_4]^T$, the original state term is $[x_1 \; x_2]$ where x_1 is the change in the reactor temperature and x_2 is the change in the mole fraction of original chemical species, and the unknown input term is $[x_3 \; x_4]$ where x_3 corresponds to the clogging of the inlet pipe and x_4 is related to the heat transfer fluid.

Objective is to obtain the unknown input estimate $[\hat{x}_3 \; \hat{x}_4]^T$ as well as the state estimate $[\hat{x}_1 \; \hat{x}_2]^T$ subject to (6) and the inequality constraints (7). The first two express the requirement that the mole fraction of original chemical species is in the interval $[0, 1]$. The next two constraints imply that the inlet flow rate and the temperature of heat transfer fluid can only decrease from their nominal values. The last constraint implies that the decrease of the inlet flow rate must be bounded.

Fig. 1. Comparison of four estimators.

The first unknown input is a step type with 0.3 decrease at $k = 100$ and the second one is also a step type with 0.5 degrees at $k = 200$. Fig. 2 and 3 show estimates for the second unknown input. Two unconstrained RHEs violate the negativity constraint in some instances. However, two constrained RHEs satisfy the negativity constraints, since they possess more information regarding unknown inputs in the form of equality constraints. It is remarkable that the RHEs with an FIR structure shown in Fig. 3 shows superior fast tracking performance to other two estimators with IIR structure shown in Fig. 2, which indicates the finite convergence time and the fast tracking ability the FIR structure filters. Therefore, in the viewpoint of both satisfying constraints and fast tracking, the constrained RHE with FIR structure will be useful for the problem of the unknown input estimation.

V. Concluding Remarks

This paper has concerned with an RHE with discrete-time linear systems subject to constraints on the estimate. In solving the optimization for every horizons, the measurement data outside the horizon is discarded and thus the arrival cost is not considered. The RHE is the FIR structure which has some good inherent properties. The proposed RHE can be represented in the simple matrix form for the unconstrained case. Via various numerical examples, it is shown how including constraints in the RHE can improve estimation performance. Especially, it is shown how the FIR structure in the RHE can improve the estimation speed in the application to the unknown input estimation on the CSTR model.

Fig. 2. RHEs with IIR structure.

Fig. 3. RHEs with FIR structure.

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