Delay-dependent Guaranteed Cost Control for Uncertain State-delayed Systems

Young Sam Lee, Oh-Kyu Kwon, and Wook Hyun Kwon

Abstract: This paper concerns delay-dependent guaranteed cost control (GCC) problem for a class of linear state-delayed systems with norm-bounded time-varying parametric uncertainties. By incorporating the free weighing matrix approach developed recently, new delay-dependent conditions for the existence of the guaranteed cost controller are presented in terms of matrix inequalities for both nominal state-delayed systems and uncertain state-delayed systems. An algorithm involving convex optimization is proposed to design a controller achieving a suboptimal guaranteed cost such that the system can be stabilized for all admissible uncertainties. Through numerical examples, it is shown that the proposed method can yield less guaranteed cost than the existing delay-dependent methods.

Keywords: Delay-dependent, guaranteed cost control, time-delay system.

1. INTRODUCTION

Since time-delay is often a source of instability in many engineering systems, considerable attention has been paid to the problem of stability analysis and controller synthesis for time-delay systems. Especially, in accordance with the advance of robust control theory, a number of robust stabilization methods have been proposed for uncertain time-delay systems.

The existing robust stabilization results for time-delay systems can be classified into two types: delay-independent stabilization [1-4] and delay-dependent stabilization [5-9]. Delay-independent stabilization provides a controller that can stabilize a system irrespective of the size of the delay. On the other hand, delay-dependent stabilization is concerned with the size of the delay and usually provides an upper bound of the delay such that the closed-loop system is stable for any delay less than the upper bound.

In addition to simple stabilization, there have been various efforts to assign certain performance criteria when designing a controller, such as quadratic cost minimization, H_∞ norm minimization, pole placement, *etc*. Among them, guaranteed cost control aims at stabilizing the systems while maintaining an adequate level of performance represented by the quadratic cost. Guaranteed cost control for time-delay systems can also be categorized into delay-independent methods [10-12] and delay-dependent methods [13-16]. The recent research trend has been focused on delay-dependent methods. In [13], delay-dependent GCC was first proposed by utilizing model transformation and Moon's inequality introduced in [7]. It was first illustrated that the delay-dependent GCC can provide even less guaranteed cost than the delay-independent GCC methods. In [14], the results of [13] were extended to the discrete-time case. The descriptor model transformation method combined with Moon's inequality was utilized in order to derive delay-dependent GCC in [14]

Recently, the free weighting matrix approach was proposed in order to overcome the conservativeness of methods involving a fixed model transformation [17]. It is expected that adopting the free weighting matrix approach for delay-dependent GCC may yield further improvement in the performance. This has motivated the work in this paper.

This paper is structured as follows: In Section 2, problem formulation and some preliminaries are given. In Section 3, guaranteed cost control for nominal time-delay systems is considered and a nonlinear minimization problem is formulated such that the suboptimal minimum of the cost is obtained. In Section 4, guaranteed cost control for uncertain time-delay systems is presented. In Section 5, numerical examples are provided and conclusions follow in Section 6.

Manuscript received December 3, 2004; revised June 23, 2005; accepted August 1, 2005. Recommended by Editorial Board member Lihua Xie under the direction of Editor-in-Chief Myung Jin Chung. This work was supported by the Inha Research Grant 32745.

Young Sam Lee and Oh-Kyu Kwon are with the School of Electrical Engineering, Inha University, 253, Nam-ku, Yonghyun-dong, Inchon City 402-751, Korea (e-mails: {lys, okkwon}@inha.ac.kr).

Wook Hyun Kwon is with the School of Electrical Engineering and Computer Science, Seoul National University, Seoul, Korea (e-mail: whkwon@cisl.snu.ac.kr).

2. PROBLEM FORMULATION

Consider uncertain linear state-delayed systems represented by

$$\begin{aligned}\dot{x}(t) &= [A + D\Delta(t)E]x(t) + [A_1 + D\Delta(t)E_1]x(t-h) \\ &\quad + [B + D\Delta(t)E_b]u(t),\end{aligned} \tag{1}$$

$$x(t) = \phi(t),\ t \in [-\bar{h}, 0], \tag{2}$$

where $x \in R^n$ is the state, $u \in R^m$ is the control input, $A, A_1, B, D, E, E_1, E_b$ are all real constant matrices of appropriate dimension and $h > 0$ is an unknown constant representing delay. $\Delta(t)$ denotes time-varying parameter uncertainties and is assumed to be of block diagonal form

$$\Delta(t) = \mathrm{diag}\{\Delta_1(t), \cdots, \Delta_r(t)\},$$

where $\Delta_i(t) \in R^{p_i \times q_i}, i = 1, \cdots, r$ are unknown real time-varying matrices satisfying

$$\Delta_i^T(t)\Delta_i(t) \le I,\ \forall t \ge 0.$$

Throughout the paper, I denotes an identity matrix of appropriate dimension. Given matrices $Q > 0$ and $R > 0$ we will consider an infinite horizon quadratic cost function represented by

$$J = \int_0^\infty [x^T(t)Qx(t) + u^T(t)Ru(t)]dt. \tag{3}$$

Associated with the cost (3), the guaranteed cost controller is defined as follows:

Definition 1: Consider the uncertain state-delayed system (1). If there exists a control law $u(t)$ and a positive scalar γ such that, for all admissible uncertainties, the closed-loop system is stable and the closed-loop value of the cost function (3) satisfies $J \le \gamma$, then γ is said to be a guaranteed cost and $u(t)$ is said to be a guaranteed cost controller for the uncertain system (1).

We are interested in designing a memoryless state-feedback controller or a delayed state-feedback controller depending on the situations as follows:

$$u(t) = \begin{cases} Kx(t) & : \text{unknown constant delay} \\ Kx(t) + K_1 x(t-h) & : \text{knwon constant delay} \end{cases}$$

which achieves as small number of γ as possible for uncertain state-delayed systems, where K, $K_1 \in R^{m \times n}$ are constant matrices.

Before moving on, we introduce a lemma necessary to take uncertainties into account.

Lemma 1 [13]**:** Let D, E, and Δ be real matrices of appropriate dimensions with $\Delta = \mathrm{diag}\{\Delta_1, \cdots, \Delta_r\}$, $\Delta_i^T \Delta_i \le I, i = 1, \cdots, r$. Then, for any real matrix $\Lambda = \mathrm{diag}\{\lambda_1 I, \cdots, \lambda_r I\} > 0$, the following inequalities will be true:

$$D\Delta E + E^T \Delta^T D^T \le D\Lambda D^T + E^T \Lambda^{-1} E. \tag{4}$$

3. GUARANTEED COST CONTROL FOR NOMINAL SYSTEMS

In the following section, instead of directly dealing with the uncertain system (1), we first consider a nominal system without uncertainties and offer stability and synthesis results. A nominal state-delayed system is represented by

$$\dot{x}(t) = Ax(t) + A_1 x(t-h) + Bu(t), \tag{5}$$

$$x(t) = \phi(t), t \in [-\bar{h}, 0]. \tag{6}$$

The following theorem states a sufficient condition for the existence of guaranteed cost control for a nominal state-delayed system (5).

Theorem 1: Given $Q > 0$ and $R > 0$, assume that there exists $L_1 > 0$, L_2, L_3, U, W, N_1, N_2, N_3, and V such that

$$\begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \bar{h}N_1 & L_1 Q^{\frac{1}{2}} & V^T R^{\frac{1}{2}} & \bar{h}L_2^T \\ * & \Gamma_{22} & \Gamma_{23} & \bar{h}N_2 & 0 & 0 & \bar{h}L_3^T \\ * & * & \Gamma_{33} & \bar{h}N_3 & 0 & 0 & 0 \\ * & * & * & \Gamma_{44} & 0 & 0 & 0 \\ * & * & * & * & -I & 0 & 0 \\ * & * & * & * & * & -I & 0 \\ * & * & * & * & * & * & -\bar{h}U \end{bmatrix} \le 0 \tag{7}$$

where

$$\begin{aligned}
\Gamma_{11} &= L_2 + L_2^T + N_1 + N_1^T + W, \\
\Gamma_{12} &= L_3 + L_2^T - (AL_1 + BV)^T + N_2^T, \\
\Gamma_{13} &= -N_1 + N_3^T, \\
\Gamma_{22} &= L_3 + L_3^T, \\
\Gamma_{23} &= -A_1 L_1 - N_2, \\
\Gamma_{33} &= -W - N_3 - N_3^T, \\
\Gamma_{44} &= -\bar{h}L_1 U^{-1} L_1,
\end{aligned}$$

then the system (5) with the control $u(t) = VL_1^{-1}x(t)$ is asymptotically stable for any constant time-delay $0 \le h \le \bar{h}$ and the cost function (3) satisfies the following bound:

$$J \le x^T(0)L_1^{-1}x(0) + \int_{-\bar{h}}^{0} x^T(\alpha)L_1^{-1}WL_1^{-1}x(\alpha)d\alpha + \int_{-\bar{h}}^{0}\int_{\beta}^{0} \dot{x}^T(\alpha)U^{-1}x(\alpha)d\alpha d\beta. \tag{8}$$

Proof: Assume the control has the form $u(t) = Kx(t)$. Then the closed-loop system is described by

$$\dot{x}(t) = A_c x(t) + A_{1c}x(t-h), \tag{9}$$

where $A_c = A + BK$ and $A_{1c} = A_1$. For $P > 0$, $S > 0$, and $Z > 0$, define

$$V(x_t) = x^T(t)Px(t) + \int_{t-h}^{t} x^T(\alpha)Sx(\alpha)d\alpha + \int_{-h}^{0}\int_{t+\beta}^{t} \dot{x}^T(\alpha)Zx(\alpha)d\alpha d\beta,$$

where x_t denotes $x_t(\theta) = x(t+\theta)$, $t \in [-\bar{h}, 0]$. Calculating the derivative of $V(x_t)$ along the solution of the system in (9) yields

$$\frac{dV(x_t)}{dt} = 2\dot{x}^T(t)Px(t) + x^T(t)Sx(t) - x^T(t-h)Sx(t-h) + h\dot{x}^T(t)Z\dot{x}(t) - \int_{t-h}^{t} \dot{x}^T(\alpha)Z\dot{x}(\alpha)d\alpha.$$

Applying the free weighting matrix approach introduced in [17], the following equations are true for any matrices Y_1, Y_2, Y_3 and T_1, T_2:

$$2[x^T(t)Y_1 + \dot{x}^T(t)Y_2 + x^T(t-h)Y_3] \times [x(t) - x(t-h) - \int_{t-h}^{t} \dot{x}(\alpha)d\alpha] = 0,$$

$$2[x^T(t)T_1 + \dot{x}^T(t)T_2] \times [\dot{x}(t) - A_c x(t) - A_{1c}x(t-h)] = 0.$$

Furthermore, it holds that

$$h\eta^T(t)X\eta(t) - \int_{t-h}^{t} \eta^T(t)X\eta(t)d\alpha = 0,$$

where

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ * & X_{22} & X_{23} \\ * & * & X_{33} \end{bmatrix} \ge 0, \quad \eta(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \\ x(t-h) \end{bmatrix}.$$

From those, we obtain

$$\frac{dV}{dt} = 2\dot{x}^T(t)Px(t) + x^T(t)Sx(t) - x^T(t-h)Sx(t-h) + h\dot{x}^T(t)Z\dot{x}(t) - \int_{t-h}^{t} \dot{x}^T(\alpha)Zx(\alpha)d\alpha$$
$$+2[x^T(t)Y_1 + \dot{x}^T(t)Y_2 + x^T(t-h)Y_3] \times [x(t) - x(t-h) - \int_{t-h}^{t} \dot{x}(\alpha)d\alpha]$$
$$+2[x^T(t)T_1 + \dot{x}^T(t)T_2][\dot{x}(t) - A_c x(t) - A_{1c}x(t-h)]$$
$$+h\eta^T(t)X\eta(t) - \int_{t-h}^{t} \eta^T(t)X\eta(t)d\alpha.$$

Therefore we have, for $\bar{h} \ge h$,

$$\frac{dV(x_t)}{dt} + x^T(t)[Q + K^T RK]x(t) \le \eta^T(t)\Xi\eta(t) - \int_{t-h}^{t} \varsigma^T(t,\alpha)\Psi\varsigma(t,\alpha)d\alpha, \tag{10}$$

where

$$\Xi = \begin{bmatrix} \bar{\Gamma}_{11} & \bar{\Gamma}_{12} & \bar{\Gamma}_{13} \\ * & \bar{\Gamma}_{22} & \bar{\Gamma}_{23} \\ * & * & \bar{\Gamma}_{33} \end{bmatrix} + \bar{h}\begin{bmatrix} X_{11} & X_{12} & X_{13} \\ * & X_{22} & X_{23} \\ * & * & X_{33} \end{bmatrix}, \tag{11}$$

$$\Psi = \begin{bmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ * & X_{22} & X_{23} & X_{24} \\ * & * & X_{33} & X_{34} \\ * & * & * & X_{44} \end{bmatrix}, \quad \varsigma(t,\alpha) = \begin{bmatrix} x(t) \\ \dot{x}(t) \\ x(t-h) \\ \dot{x}(\alpha) \end{bmatrix},$$

$$\bar{\Gamma}_{11} = S + K^T RK + Y_1 + Y_1^T - T_1 A_c - A_c^T T_1^T,$$
$$\bar{\Gamma}_{12} = P + Y_2^T + T_1 - A_c^T T_2^T,$$
$$\bar{\Gamma}_{13} = Y_3^T - Y_1 - T_1 A_{1c},$$
$$\bar{\Gamma}_{22} = \bar{h}Z + T_2 + T_2^T,$$
$$\bar{\Gamma}_{23} = -Y_2 - T_2 A_{1c},$$
$$\bar{\Gamma}_{33} = -S - Y_3 - Y_3^T.$$

From (10), we see that $\Xi \le 0$ and $\Psi \ge 0$ guarantee the asymptotic stability of the closed-loop system because we obtain

$$\frac{dV(x_t)}{dt} \le -x^T(t)[Q + K^T RK]x(t), \tag{12}$$

which implies that $V(x_t)$ is a Lyapunov-Krasovskii functional. Furthermore, integration of (12) from 0 to ∞ yields

$$V(x_\infty) - V(x_0) \le -\int_0^\infty x^T(t)[Q + K^T RK]x(t)dt.$$

Because $V(x_\infty) = 0$ from asymptotic stability, we have

$$V(x_\infty) \ge \int_0^\infty x^T(t)[Q + K^T RK]x(t)dt = \int_0^\infty [x^T(t)Qx(t) + u^T(t)Ru(t)]dt = J.$$

Therefore we have, for $\bar{h} \ge h$,

$$J \le V(x_0) = x^T(0)Px(0) + \int_{-h}^{0} x^T(\alpha)Sx(\alpha)d\alpha + \int_{-h}^{0}\int_{\beta}^{0} \dot{x}(\alpha)Z\dot{x}(\alpha)d\alpha d\beta \quad (13)$$

$$\le x^T(0)Px(0) + \int_{-\bar{h}}^{0} x^T(\alpha)Sx(\alpha)d\alpha + \int_{-\bar{h}}^{0}\int_{\beta}^{0} \dot{x}(\alpha)Z\dot{x}(\alpha)d\alpha d\beta. \quad (14)$$

The remaining part of the proof is to show that the satisfaction of (7) guarantees $\Xi \le 0$ and $\Psi \ge 0$ and the right-hand side of (14) is equal to the right-hand side of (8). If we select $Z > 0$ and X such that

$$X = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} Z^{-1} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}^T,$$

then it is guaranteed that $X \ge 0$ and $\Psi \ge 0$. From Schur complement, $\Xi \le 0$ is equivalent to

$$\begin{bmatrix} \bar{\Gamma}_{11} & \bar{\Gamma}_{12} & \bar{\Gamma}_{13} & \bar{h}Y_1 \\ * & \bar{\Gamma}_{22} & \bar{\Gamma}_{23} & \bar{h}Y_2 \\ * & * & \bar{\Gamma}_{33} & \bar{h}Y_3 \\ * & * & * & -\bar{h}Z_1 \end{bmatrix} \le 0. \quad (15)$$

Define

$$\begin{bmatrix} P & 0 \\ T_2^T & T_3^T \end{bmatrix}^{-1} = L = \begin{bmatrix} L_1 & 0 \\ L_2 & L_3 \end{bmatrix}. \quad (16)$$

Pre- and post-multiply (15) by $\mathrm{diag}\{L^T, L_1, L_1\}$ and $\mathrm{diag}\{L, L_1, L_1\}$. Introduce some change of variables such that

$$W = L_1 S L_1, U = Z^{-1}, V = KL_1,$$
$$\begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = L^T \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} L_1, N_3 = L_1 Y_3 L_1. \quad (17)$$

Then (15) is equivalently changed to (7) from Schur complement. From (16) and (17), we have

$$P = L_1^{-1}, S = L_1^{-1} W L_1^{-1}, Z = U^{-1}. \quad (18)$$

Substituting (18) into (14), we obtain the relation in (8). This completes the proof. □

It is noted that the inequality (7) is not an LMI condition because of the term $-\bar{h}L_1U^{-1}L_1$. Theorem 1 considers the case that the delay is constant but unknown. However, if the delay length is constant and exactly known, we can use the delay information and construct delayed state-feedback control as in the following corollary:

Corollary 1: Given $Q > 0$ and $R > 0$, assume that there exist $L_1 > 0, L_2, L_3, U, W, N_1, N_2, N_3, V,$ and V_1 such that

$$\begin{bmatrix} \tilde{\Gamma}_{11} & \tilde{\Gamma}_{12} & \tilde{\Gamma}_{13} & hN_1 & \tilde{\Gamma}_{15} & \tilde{\Gamma}_{16} & hL_2^T \\ * & \tilde{\Gamma}_{22} & \tilde{\Gamma}_{23} & hN_2 & 0 & 0 & hL_3^T \\ * & * & \tilde{\Gamma}_{33} & hN_3 & 0 & \tilde{\Gamma}_{36} & 0 \\ * & * & * & \tilde{\Gamma}_{44} & 0 & 0 & 0 \\ * & * & * & * & -I & 0 & 0 \\ * & * & * & * & * & -I & 0 \\ * & * & * & * & * & * & -hU \end{bmatrix} \le 0 \quad (19)$$

where

$$\tilde{\Gamma}_{11} = L_2 + L_2^T + N_1 + N_1^T + W,$$
$$\tilde{\Gamma}_{12} = L_3 + L_2^T - (AL_1 + BV)^T + N_2^T,$$
$$\tilde{\Gamma}_{13} = -N_1 + N_3^T, \tilde{\Gamma}_{15} = L_1 Q^{\frac{1}{2}}, \tilde{\Gamma}_{16} = V^T R^{\frac{1}{2}},$$
$$\tilde{\Gamma}_{22} = L_3 + L_3^T, \tilde{\Gamma}_{23} = -(A_1 L_1 + BV_1) - N_2,$$
$$\tilde{\Gamma}_{33} = -W - N_3 - N_3^T, \tilde{\Gamma}_{36} = V_1^T R^{\frac{1}{2}},$$
$$\tilde{\Gamma}_{44} = -hL_1U^{-1}L_1,$$

then the system (5) with the control $u(t) = VL_1^{-1}x(t) + V_1L_1^{-1}x(t-h)$ is asymptotically stable for known constant time-delay h and the cost function (3) satisfies the following bound:

$$J \le x^T(0)L_1^{-1}x(0) + \int_{-h}^{0} x^T(\alpha)L_1^{-1}WL_1^{-1}x(\alpha)d\alpha + \int_{-h}^{0}\int_{\beta}^{0} \dot{x}^T(\alpha)U^{-1}x(\alpha)d\alpha d\beta. \quad (20)$$

Proof: In case that the delay length is precisely known, we assume the control $u(t) = Kx(t) + K_1x(t-h)$.

Then the closed-loop system reduces to (9) with $A_c = A + BK$ and $A_{1c} = A_1 + BK_1$. The remaining proof procedure is straightforward from the proof of Theorem 1. □

Given $\bar{h}$, in order to obtain the controller $u(t) = VL_1^{-1}(t)$, which achieves the least guaranteed cost value γ^*, we have to solve the following minimization problem:

$$\text{Minimize } J_1 + J_2 + J_3 \quad \text{subject to } L_1 > 0 \text{ and (7)} \quad (21)$$

where

$$J_1 = x^T(0)L_1^{-1}x(0),$$
$$J_2 = \int_{-\bar{h}}^{0} x^T(\alpha)L_1^{-1}WL_1^{-1}x(\alpha)d\alpha,$$
$$J_3 = \int_{-\bar{h}}^{0}\int_{-\beta}^{0} \dot{x}^T(\alpha)U^{-1}\dot{x}(\alpha)d\alpha d\beta.$$

However, it is noted that the condition (7) is no more an LMI condition because of the term $-\bar{h}L_1U^{-1}L_1$. In addition, the term J_2 is not a convex function of L_1 and W. As a result, unfortunately, we cannot find in general the global minimum of the above minimization problem using a convex optimization algorithm. However, utilizing the idea in [7, 9], we can obtain a guaranteed cost controller achieving a suboptimal guaranteed cost, say γ_{so}, using the iterative algorithm as presented next.

First, we define a new variable M such that $L_1U^{-1}L_1 > M$ and replace the condition (7) with

$$\begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \bar{h}N_1 & L_1Q^{\frac{1}{2}} & V^TR^{\frac{1}{2}} & \bar{h}L_2^T \\ * & \Gamma_{22} & \Gamma_{23} & \bar{h}N_2 & 0 & 0 & \bar{h}L_3^T \\ * & * & \Gamma_{33} & \bar{h}N_3 & 0 & 0 & 0 \\ * & * & * & -\bar{h}M & 0 & 0 & 0 \\ * & * & * & * & -I & 0 & 0 \\ * & * & * & * & * & -I & 0 \\ * & * & * & * & * & * & -\bar{h}U \end{bmatrix} \le 0 \tag{22}$$

and

$$L_1U^{-1}L_1 > M. \tag{23}$$

Since $L_1U^{-1}L_1 > M$ is equivalent to $L_1^{-1}UL_1^{-1} < M_1^{-1}$, the condition (23) is equal to

$$\begin{bmatrix} M^{-1} & L_1^{-1} \\ * & U^{-1} \end{bmatrix} > 0. \tag{24}$$

Then, by introducing a new variable $\bar{M}, \bar{L}_1$ and $\bar{U}$, the condition (24) can be replaced by

$$\begin{bmatrix} \bar{M} & \bar{L}_1 \\ * & \bar{U} \end{bmatrix}, \ \bar{M} = M^{-1}, \ \bar{L}_1 = L_1^{-1}, \ \bar{U} = U^{-1}. \tag{25}$$

It is noted that the satisfaction of the conditions (22) and (25) guarantees that the condition (7) is satisfied. Let us derive the upper bounds on the cost function J_1, J_2, and J_3. For this purpose, let's denote

$$\Psi = \int_{-\bar{h}}^{0} x(s)x^T(s)ds,$$
$$\Phi = \int_{-\bar{h}}^{0}\int_{\beta}^{0} \dot{x}(\alpha)\dot{x}^T(\alpha)d\alpha d\beta.$$

Because $\Phi \ge 0$, it always can be factorized as $\Phi = \Pi\Pi^T$. We first start with the upper bound on J_1. Assume that there exists $\alpha > 0$ which satisfies

$$x^T(0)L_1^{-1}x(0) \le \alpha. \tag{26}$$

From the Schur complement, (26) is equivalent to

$$\begin{bmatrix} \alpha & x^T(0) \\ * & L_1 \end{bmatrix} \ge 0. \tag{27}$$

To derive the upper bound on J_2, introduce a new variable $\Omega = \Omega^T$ such that

$$L_1^{-1}WL_1^{-1} < \Omega. \tag{28}$$

By the Schur complement, (28) is equivalent to

$$\begin{bmatrix} \Omega & L_1^{-1} \\ * & W^{-1} \end{bmatrix} > 0. \tag{29}$$

Recalling $L_1^{-1} = \bar{L}_1$ and introducing a new variable $\bar{W} = W^{-1}$, the condition (29) can be replaced by

$$\begin{bmatrix} \Omega & \bar{L}_1 \\ * & \bar{W} \end{bmatrix} > 0, \ \bar{W} = W^{-1}. \tag{30}$$

Assuming (30), we can conclude that the following relation holds:

$$J_2 \le tr(\Omega\Psi).$$

Recalling $tr(AB) = tr(BA)$, we know that the following relation holds for J_3:

$$J_3 = tr(\Pi\Pi^TU^{-1}) = tr(\Pi U^{-1}\Pi).$$

Let us introduce a new matrix variable $\Sigma = \Sigma^T$ such that

$$\Pi U^{-1}\Pi^T \le \Sigma. \tag{31}$$

By the Schur complement, (31) is equivalent to

$$\begin{bmatrix} \Sigma & \Pi \\ * & U \end{bmatrix} \ge 0. \tag{32}$$

Under the condition (32), J_3 satisfies $J_3 \le tr(\Sigma)$.

It is noted that $J = J_1 + J_2 + J_3 \le \alpha + tr(\Sigma) + tr(\Omega\Psi)$

is true under the conditions (27), (30), and (32). For some constant $\gamma > 0$, assume

$$\alpha + tr(\Sigma) + tr(\Omega\Psi) \le \gamma . \tag{33}$$

Combining those facts derived above, we can construct a feasibility problem for given $\bar{h}$ and γ as follows:

$$\begin{aligned} &\text{Find } L_1, \bar{L}_1, L_2, L_3, N_1, N_2, N_3, V, W, U, M, \\ &\quad \bar{W}, \bar{U}, \bar{M}, \Omega, \Sigma, \alpha \\ &\text{subject to } L_1 > 0 \text{ and } (22), (25), (27), \\ &\qquad (30), (32), (33). \end{aligned} \tag{34}$$

Given $\bar{h}$ and γ, if the above problem has a solution, we can say that there exists a controller $u(t) = VL_1^{-1}(t)$ which guarantees that the cost function (3) is less than γ. It is noted that conditions (25) and (30) still include nonlinear conditions, e.g. $\bar{L}_1 = L_1^{-1}$ and $\bar{U} = U^{-1}$. However, using the idea introduced in [7, 9], the feasibility problem in (34) can be converted to the following nonlinear minimization problem involving LMI conditions:

$$\begin{aligned} &\text{Minimize } tr(U\bar{U} + L_1\bar{L}_1 + M\bar{M} + W\bar{W}) \\ &\text{subject to} \\ &\begin{cases} L_1 > 0, (22), (27), (32), (33) \\ \begin{bmatrix} \bar{M} & \bar{L}_1 \\ * & \bar{U} \end{bmatrix} > 0, \begin{bmatrix} \Omega & \bar{L}_1 \\ * & \bar{W} \end{bmatrix} > 0, \\ \begin{bmatrix} L_1 & I \\ * & \bar{L}_1 \end{bmatrix} \ge 0, \begin{bmatrix} M & I \\ * & \bar{M} \end{bmatrix} \ge 0, \\ \begin{bmatrix} U & I \\ * & \bar{U} \end{bmatrix} \ge 0, \begin{bmatrix} W & I \\ * & \bar{W} \end{bmatrix} \ge 0. \end{cases} \end{aligned} \tag{35}$$

If the solution of the above minimization problem is $4n$, that is, $tr(U\bar{U} + L_1\bar{L}_1 + M\bar{M} + W\bar{W}) = 4n$, we can say from Theorem 1 that system (5) with the control $u(t) = VL^{-1}x(t)$ is asymptotically stable with the guaranteed cost γ. Even though the above minimization problem is also a nonlinear one, an iterative solution procedure similar to those proposed in [7, 9] can also be derived as follows:

Algorithm

1) Given $\bar{h}$, choose a sufficiently large initial γ such that there exists a feasible solution to LMI conditions in (35). Set $\gamma_{so} = \gamma$.

2) Find feasible set ($L_1^0, \bar{L}_1^0, L_2^0, L_3^0, N_1^0, N_2^0, N_3^0, V^0, W^0, U^0, M^0, \bar{W}^0, \bar{U}^0, \bar{M}^0, \Omega^0, \Sigma^0, \alpha^0$) satisfying LMIs in (35). $k = 1$.

3) Solve the following LMI problem for the variables ($L_1, \bar{L}_1, L_2, L_3, N_1, N_2, N_3, V, W, U, M, \bar{W}, \bar{U}, \bar{M}, \Omega, \Sigma, \alpha$):

$$\begin{aligned} &\text{Minimize } tr(U^k\bar{U} + L_1^k\bar{L}_1 + M^k\bar{M} + W^k\bar{W} + \bar{U}^k U \\ &\qquad + \bar{L}_1^k L_1 + \bar{M}^k M + \bar{W}^k W) \end{aligned}$$

subject to LMIs in (35).

Set $U^{k+1} = U, \bar{U}^{k+1} = \bar{U}, L_1^{k+1} = L_1, \bar{L}_1^{k+1} = \bar{L}_1, \bar{M}^{k+1} = \bar{M}, W^{k+1} = W, \bar{W}^{k+1} = \bar{W}$.

4) If the conditions (23) and (28) are both satisfied, then set $\gamma_{so} = \gamma$ and return to Step 2 after decreasing γ to some extent. If the conditions (23) and (28) are not satisfied within a specified number of iterations, say $k_{\max}$, then exit. Otherwise, set $k = k + 1$ and go to Step 3.

The above algorithm gives a guaranteed cost controller $u(t) = VL_1^{-1}x(t)$ and the corresponding suboptimal guaranteed cost γ_{so}. Later, in Section 5, we will illustrate via numerical example that the above algorithm can provide quite satisfactory results.

Remark 1: For known constant delay, we can also construct an algorithm yielding the controller $u(t) = VL_1^{-1}x(t) + V_1L_1^{-1}x(t-h)$ and γ_{so} following the similar procedure described above. It is expected that for known constant delay, we can obtain a smaller value of γ_{so} than for unknown delay because we use the delay information in the controller.

4. GUARANTEED COST CONTROL FOR UNCERTAIN SYSTEMS

In this section, we extend the conditions obtained in Section 3 to robust conditions for the uncertain systems (1). The following theorem states the sufficient condition for the existence of guaranteed cost control for uncertain time-delay systems.

Theorem 2: Given $Q > 0$ and $R > 0$ assume that there exist $L_1 > 0$, L_2, L_3, U, W, N_1, N_2, N_3, V, and $\Lambda = \text{diag}\{\lambda_1 I, \cdots, \lambda_r I\}$ such that

$$\begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Theta_{14} & \Theta_{15} & \Theta_{16} & \Theta_{17} & \Theta_{18} \\ * & \Theta_{22} & \Gamma_{23} & \Theta_{24} & 0 & 0 & \Theta_{27} & 0 \\ * & * & \Gamma_{33} & \Theta_{34} & 0 & 0 & 0 & \Theta_{38} \\ * & * & * & \Theta_{44} & 0 & 0 & 0 & 0 \\ * & * & * & * & -I & 0 & 0 & 0 \\ * & * & * & * & * & -I & 0 & 0 \\ * & * & * & * & * & * & \Theta_{77} & 0 \\ * & * & * & * & * & * & * & -\Lambda \end{bmatrix} \le 0, \tag{36}$$

where

$$\Theta_{14} = \bar{h}N_1,\ \Theta_{15} = L_1 Q^{\frac{1}{2}},\ \Theta_{16} = V^T R^{\frac{1}{2}},$$
$$\Theta_{17} = \bar{h}L_2^T, \Theta_{18} = (EL_1 + E_b V)^T,$$
$$\Theta_{22} = \Gamma_{22} + D\Lambda D^T,\ \Theta_{24} = \bar{h}N_2, \Theta_{27} = \bar{h}L_3^T,$$
$$\Theta_{34} = \bar{h}N_{3,}\ \Theta_{38} = (E_1 L_1)^T,$$
$$\Theta_{44} = -\bar{h}L_1 U^{-1} L_1,\ \Theta_{77} = -\bar{h}U,$$

then the uncertain system (5) with the control $u(t) = VL_1^{-1}x(t)$ is asymptotically stable for any constant time-delay $0 \le 0 \le \bar{h}$ and the cost function (3) satisfies the following bound:

$$J \le x^T(0)L_1^{-1}x(0) + \int_{-\bar{h}}^{0} x^T(\alpha)L_1^{-1}WL_1^{-1}x(\alpha)d\alpha + \int_{-\bar{h}}^{0}\int_{\beta}^{0} \dot{x}^T(\alpha)U^{-1}x(\alpha)d\alpha d\beta. \tag{37}$$

Proof: It is sufficient for the proof of Theorem 2 to show that (7) is still satisfied even with A, A_1, and B replaced by $A + D\Delta E$, $A_1 + D\Delta E_1$, and $B + D\Delta E_b$, respectively. Define the matrix in the left side of '$\le$' in (7) to be Γ. Then, the condition (7) with A, A_1, and B replaced by $A + D\Delta E$, $A_1 + D\Delta E_1$, and $B + D\Delta E_b$, respectively, is written as

$$\Gamma + \bar{D}\Delta\bar{E} + \bar{E}^T\Delta^T\bar{D}^T < 0, \tag{38}$$

where

$$\bar{D} \triangleq -[0\ \ D^T\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0]^T,$$
$$\bar{E} \triangleq [EL_1 + E_b V\ \ 0\ \ E_1 L_1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0].$$

According to (4) in Lemma 1, (38) holds if there exists $\Lambda = \mathrm{diag}\{\lambda_1 I, \cdots, \lambda_r I\} > 0$ such that

$$\Gamma + \bar{D}\Lambda\bar{D} + \bar{E}^T\Lambda^{-1}\bar{E} < 0. \tag{39}$$

By the Schur complement, (39) is equivalent to (36). This completes the proof. □

Similarly to the nominal system case, we can derive results for known constant time-delay h.

5. EXAMPLES

In this section, we provide two numerical examples in order to illustrate that the proposed method is less conservative than the existing results.

Example 1: Consider the uncertain time-delay system represented by

$$\dot{x}(t) = \begin{bmatrix} q(t) & 1+q(t) \\ 1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} r(t) & r(t) \\ 0.1 & 0.1 \end{bmatrix} x(t-h) + \begin{bmatrix} 0 \\ 1+s(t) \end{bmatrix} u(t),$$

where $h = 1$, $x_1(t) = 0.5e^{\frac{t}{2}}$, and $x_2(t) = -e^{-\frac{t}{2}}$, for $t \in [-1, 0]$. $q(t),\ r(t),\ s(t)$ are scalar uncertain parameters satisfying the bound $|q(t)| \le 0.5$, $|r(t)| \le 0.5$, and $|s(t)| \le 0.5$. It is assumed that $Q = I$ and $R = 1$ in the cost function (3). For this system, we obtain

$$A = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix},\ A_1 = \begin{bmatrix} 0 & 0 \\ -0.1 & -0.1 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$D = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}^T, E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}^T,$$
$$E_b = [0\ \ 0\ \ 1]^T,\ \Delta(t) = 2 \times \mathrm{diag}\{q(t), r(t), s(t)\}.$$

This system is delay-independently stabilizable. Therefore, we first tested the delay-independent GCC proposed in [12]. For the purpose of comparison, we also tested the method proposed in [13]. Furthermore, the continuous-time counterpart of the method in [14] was derived and tested. Table 1 compares the costs and the controllers obtained from those four methods. It clearly shows that the proposed method yields much less cost than the delay-independent method. Furthermore, the proposed method outperforms the two other existing delay-dependent GCC methods.

Example 2: Consider the uncertain time-delay ystems given in [6].

Table 1. Comparison of the obtained guaranteed cost (Example 1).

Method	Cost	Controller matrix, K
Method of [12]	20.883	$[-38.21\ \ -14.46]$
Method of [13]	19.5	$[-40.26\ \ -16.10]$
Continuous time results of [14]	14.42	$[-17.6\ \ -9.31]$
Proposed method	13.8	$[-18.7\ \ -10.43]$

Table 2. Comparison of the obtained guaranteed cost (Example 2).

Method	Cost	Controller matrix, K
Method of [12]	×	Infeasible
Method of [13]	×	Infeasible
Continuous time results of [14]	4.55	$[0.0986\ \ -13.42]$
Proposed method	4.2	$[0.0579\ \ -2.93]$

$$\dot{x}(t) = [A + \Delta A(t)]x(t) + [A_1 + \Delta A_1(t)]x(t-h) + Bu(t),$$

where

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -2 & -0.5 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and $\Delta A(t)$ and $\Delta A_1(t)$ are uncertain matrices satisfying

$$\|\Delta A(t)\| \le 0.2, \ \|\Delta A_1(t)\| \le 0.2 .$$

For the above system, we have

$$D = \begin{bmatrix} 0.2 & 0 & 0.2 & 0 \\ 0 & 0.2 & 0 & 0.2 \end{bmatrix}, E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}^T,$$

$$E_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^T, E_b = [0 \ 0 \ 0 \ 0]^T,$$

$$\Delta(t) = \begin{bmatrix} \Delta_1(t) & 0 \\ 0 & \Delta_2(t) \end{bmatrix},$$

where $\Delta_1(t), \Delta_2(t) \in R^{2\times 2}$ such that $\Delta_i^T(t)\Delta_i(t) \le I, i = 1,2$. It is noted that the system is not delay-independently stabilizable. Assume that the initial conditions are $x_1(t) = e^{t+1}$ and $x_2(t) = 0$ for $t \in [-h, 0]$ and $h = 0.37$. We chose $Q = I$, $R = 1$. Table 2 compares the costs and controllers obtained for this system. Because the system is not delay-independently stabilizable, the method proposed in [12] is not applicable in this case. Furthermore, it turned out that the method in [13] is not applicable either due to infeasibility. It turned out that the method in [14] and the proposed method were applicable. Table 2 shows that the proposed method has better performance than the method of [14] in this case.

6. CONCLUSIONS

In this paper, we have presented a new delay-dependent guaranteed cost control for uncertain time-delay systems. While the existing delay-dependent GCC methods adopt the model transformation method combined with Moon's inequality, the proposed method is based on the recently developed free weighting matrix approach in order to overcome the conservativeness of methods involving a fixed model transformation. First, guaranteed cost control for nominal delay systems was considered. Second, it was extended to the case of delay systems with parametric uncertainties. An algorithm involving convex optimization was also proposed to construct a controller with a suboptimal guaranteed cost such that the system can be stabilized for all admissible uncertainties. It was shown by numerical examples that the proposed delay-dependent guaranteed cost control method can provide even less guaranteed cost than the delay-independent method. Furthermore, it turned out that the proposed method outperforms all existing delay-dependent guaranteed cost control methods.

REFERENCES

[1] S. Phoojaruenchanachai and K. Furuta, "Memoryless stabilization of uncertain time-varying state delays," *IEEE Trans. on Automatic Control*, vol. 37, no. 7, pp. 1022-1026, 1992.

[2] M. S. Mahmoud and N. F. Al-Muthairi, "Quadratic stabilization of continuous time systems with state-delay and norm-bounded time-varying uncertainties," *IEEE Trans. on Automatic Control*, vol. 39, no. 10, pp. 2135-2139, 1994.

[3] J. H. Lee, S. W. Kim, and W. H. Kwon, "Memoryless H_∞ controllers for state delayed systems," *IEEE Trans. on Automatic Control*, vol. 39, pp. 159-162, 1994.

[4] J. H. Kim, E. T. Jeung, and H. B. Park, "Robust control for parameter uncertain delay systems in state and control input," *Automatica*, vol. 32, no. 9, pp. 1337-1339, 1996.

[5] X. Li and C. E. D. Souza, "Criteria for robust stability and stabilization of uncertain linear systems with state delays," *Automatica*, vol. 33, no. 9, pp. 1657-1662, 1997.

[6] X. Li and C. E. D. Souza, "Delay-dependent robust stability and stabilization of uncertain linear delay systems: A linear matrix inequality approach," *IEEE Trans. on Automatic Control*, vol. 42, no. 8, pp. 1144-1148, 1997.

[7] Y. S. Moon, P. G. Park, W. H. Kwon, and Y. S. Lee, "Delay-dependent robust stabilization of uncertain state-delayed systems," *International Journal of Control*, vol. 74, no. 14, pp. 1447-1455, 2001.

[8] E. Fridman and U. Shaked, "A descriptor system approach to H_∞ control of linear time-delay systems," *IEEE Trans. on Automatic Control*, vol. 47, no. 2, pp. 253-270, 2002.

[9] Y. S. Lee, Y. S. Moon, W. H. Kwon, and P. G. Park, "Delay-dependent robust H_∞ control for uncertain systems with a state-delay," *Automatica*, vol. 40, pp. 65-72, 2004.

[10] S. O. R. Moheimani and I. R. Petersen, "Optimal quadratic guaranteed cost control of a class of uncertain time-delay systems," *IEE Proc.-Control Theory Appl*, vol. 144, no. 2, pp. 183-188, 1997.

[11] S. H. Esfahani and S. O. R. Moheimani, "LMI approach to suboptimal guaranteed cost control for uncertain time-delay systems," *IEE Proc.-*

Control Theory Appl, vol. 145, no. 6, pp. 491-498, 1998.

[12] L. Yu and J. Chu, "An LMI approach to guaranteed cost control of linear uncertain time-delay system," *Automatica*, vol. 35, pp. 1155-1159, 1999.

[13] Y. S. Lee, Y. S. Moon, and W. H. Kwon, "Delay-dependent guaranteed cost control for uncertain state-delayed systems," *Proc. of American Control Conference*, pp. 3376-3381, Arlington, USA, 2001.

[14] W. H. Chen, Z. H. Guan, and X. M. Lu, "Delay-dependent guaranteed cost control for uncertain discrete-time systems with delay," *IEE Proc.-Control Theory Appl.*, vol. 150, no. 4, pp. 412-416, 2003.

[15] W. H. Chen, Z. H. Guan, and X. M. Lu, "Delay-dependent output feedback guaranteed cost control for uncertain time-delay systems," *Automatica*, vol. 40, no. 7, pp. 1263-1268, 2004.

[16] W. H. Chen and X. M. Lu, "Delay-dependent guaranteed cost control for uncertain discrete-time systems with both state and input delays," *Journal of the Franklin Institute*, vol. 341, no. 5, pp. 419-430, 2004.

[17] Y. He, M. Wu, J. H. She, and G. P. Liu, "Delay-dependent robust stability criteria for uncertain neutral systems with mixed delays," *System and Controller Letters*, vol. 51, pp. 57-65, 2004.

Young Sam Lee received the B.S. and M.S. degrees in Electrical Engineering from Inha University, Inchon, Korea in 1997 and 1999, respectively. He received the Ph.D. at the School of Electrical Engineering and Computer Science from Seoul National University, Seoul, Korea, in 2003. His research interests include time delay systems, receding horizon control, signal processing, and embedded systems. He is currently with the School of Electrical Engineering, Inha University, Inchon, Korea.

Oh-Kyu Kwon received the B.S., M.S., and Ph.D. degrees in Electrical Engineering from Seoul National University, Seoul, Korea in 1978, 1980 and 1985, respectively. His research interests include receding horizon control, robust control, and signal processing. He is currently with the School of Electrical Engineering, Inha University, Inchon, Korea.

Wook Hyun Kwon received the B.S. and M.S. degrees in Electrical Engineering from Seoul National University, Seoul, Korea, in 1966 and 1972, respectively. He received the Ph.D. degree from Brown University, Providence, RI, in 1975. His main research interests are currently multivariable robust and predictive controls, statistical signal processing, discrete event systems, and industrial networks. He is currently with the School of Electrical Engineering and Computer Science, Seoul National University, Seoul, Korea.