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Constrained receding horizon controls for nonlinear time-delay systems

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Abstract This paper proposes a constrained receding horizon control (RHC) for a nonlinear time-delay system with input and state delays. The control law is obtained by minimizing a receding horizon cost function with weighting functions of inputs and states on the end portion of the horizon. For stability, a general condition on the weighting functions is presented and its feasibility is illustrated via a certain type of nonlinear time-delay systems. In order to deal with input and state constraints, an invariant set is obtained, where the trajectories of the inputs and the states satisfy given constraints and stay forever under some conditions. It is shown in a numerical example that the proposed RHC guarantees the closed-loop stability for nonlinear time-delay systems while meeting the constraints.

Keywords Receding horizon control (RHC) · Nonlinear time-delay systems · Input and state constraints · Terminal weighting function

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1 Introduction

For a long time, receding horizon control (RHC), also known as model predictive control (MPC) has been well known to be an effective strategy in industry as well as academy. Basically, the popularity of the RHC stems from the fact that optimization is taken over the finite time to obtain the controls. As an optimization based control, the RHC offers a good performance in comparison with other controls [1–5]. The RHC can also handle input and state constraints or somewhat complicated hybrid systems through the finite time optimization [6, 7]. Furthermore, an extension of the RHC to nonlinear systems is so tractable that it has attracted much attention recently as a practical control scheme [8–12].

Time-delays in inputs and states arise frequently in engineering and science problems: chemical process, communication and network systems, transportation systems, and biological systems [13–15]. On the optimal controls for such time-delay systems, there are some researches presented in 1960 to 1980 [16–19]. However, these had not been developed any further because of the difficulty in the computation and the implementation. Recently, the advancement of computer technologies enables us to implement such complicated optimal controls for time-delay systems and compute them within a reasonable time [20, 21]. Based on such optimal controls for linear time-delay systems, the authors proposed the unconstrained RHCs for input-delayed systems [22] and state-delayed systems [23, 24] individually. Since nowadays computing power can afford to obtain the optimal controls even for general constrained nonlinear time-delay systems with both input and state delays, it would be meaningful to obtain a stabilizing constrained RHC for such a system. As mentioned before, the RHC could be also a good choice for nonlinear time-delay systems due to its good extensibility to nonlinear systems.

In this paper, we propose a constrained receding horizon control with the guaranteed closed-loop stability for nonlinear time-delay systems with input and state delays. For stability, a new cost function with weighting functions of inputs and states on the end portion of the horizon is introduced. As a generalization of the concepts for delay-free systems, a cost monotonicity condition and an invariant set for a nonlinear time-delay system are introduced from the proposed cost function. Specially, a functional set is introduced as an invariant set, which would be a generalized version of an existing Euclidean ellipsoid set for delay free systems. In order to handle constraints, some conditions on inputs and states on the end portion of the horizon are also proposed. The feasibility and the performance are illustrated through a numerical example for a certain type of nonlinear systems. All simulation files are available from the website [25].

This paper is structured as follows: In Sect. 2, the problem is formulated with the introduction of some assumptions and notations. In Sect. 3, the condition for nonincreasing monotonicity of the optimal cost is presented, from which an invariant set is introduced. In Sect. 4, the cost monotonicity condition and the invariant set are put together to achieve the stability of the RHC. In Sect. 5, a special class of nonlinear time-delay systems is introduced for the illustration of the proposed RHC. In Sect. 6, a numerical example is given and the conclusions will be drawn in Sect. 7.

2 Problem formulation

Consider the following nonlinear time-delay system:

$$\begin{aligned} &\dot{x}(t) = f\big(x(t), x(t-h_x), u(t), u(t-h_u)\big), \\ &x(t) = \phi(t), \quad t \in [-h_x, 0], \\ &u(t) = \psi(t), \quad t \in [-h_u, 0], \end{aligned} \tag{1}$$

where $x(\cdot) \in \Re^n$ is the state, $u(\cdot) \in \Re^m$ is the control input, h_x and h_u are the state and input delay size, respectively, and $\phi(\cdot) \in \mathbf{C}_n[-h_x, 0]$ and $\psi(\cdot) \in \mathbf{C}_m[-h_u, 0]$ are the given initial values for the state and the input, respectively. It is assumed that the input and the state are restricted to some regions

$$u(\cdot) \in \mathcal{U} \subset \Re^n, \qquad x(\cdot) \in \mathcal{X} \subset \Re^m, \tag{2}$$

where $\mathcal{U}$ and $\mathcal{X}$ are compact sets including the origin. The controls should steer the states to the origin while satisfying the constraints (2).

In order to obtain the RHC, we will first consider the following finite horizon cost function:

$$\begin{aligned} J(x, u, t_0, t_1) = &\int_{t_0}^{t_1} \big\{ l_1\big(x(\tau)\big) + l_2\big(u(\tau)\big) \big\}\, d\tau \\ &+ J_F(x_{t_1}, u_{t_1}), \end{aligned} \tag{3}$$

where t_0 is the initial time, t_1 is the final time, $l_1(\cdot)$ and $l_2(\cdot)$ are positive functions of the state and the input, respectively, x_t and u_t denotes $x(t+\theta)$, $\theta \in [-h_x, 0]$, and $u(t+\theta)$, $\theta \in [-h_u, 0]$, respectively, and $J_F(\cdot,\cdot)$ is a terminal weighting cost function given by

$$\begin{aligned} J_F(x_{t_1}, u_{t_1}) = &\, g_1\big(x(t_1)\big) + \int_{t_1-h_x}^{t_1} g_2\big(x(\tau)\big)\, d\tau \\ &+ \int_{t_1-h_u}^{t_1} g_3\big(u(\tau)\big)\, d\tau, \end{aligned} \tag{4}$$

with some positive functions $g_1(\cdot)$, $g_2(\cdot)$, and $g_3(\cdot)$. It is assumed that $\alpha_1(\|x\|) \le l_1(x) \le \alpha_2(\|x\|)$, $\alpha_3(\|u\|) \le l_2(u) \le \alpha_4(\|u\|)$, $\alpha_5(\|x\|) \le g_1(x) \le \alpha_6(\|x\|)$, $\alpha_7(\|x\|) \le g_2(x) \le \alpha_8(\|x\|)$, and $\alpha_9(\|x\|) \le g_3(x) \le \alpha_{10}(\|x\|)$ with continuous, positive-definite, and strictly increasing functions $\alpha_i\colon [0,\infty) \to [0,\infty)$, $i = 1, \ldots, 10$, satisfying $\alpha_i(0) = 0$. It is noted that the terminal weighting cost function (4) has three terms related to the final state at the time t_1, and the states and the inputs on the end portions $[t_1 - h_x, t_1]$ and $[t_1 - h_u, t_1]$ of the horizon. It is shown later on that these terms play a key role in designing the RHC that guarantees the closed-loop stability. Even though two terms in (4) of the paper do not have much physical meaning, they make cost functions become Lyapunov functions.

The finite horizon optimal control will be obtained to minimize the cost function (3) with the initial time t_0 and the final time t_f while satisfying the constraints

(2) and some conditions discussed in the next section. The RHC with the horizon size T can be then obtained by replacing t_0 and t_f with the current time t and $t+T$, respectively. The stability of the proposed RHC depends on the choice of final weighting functions $g_1(\cdot)$, $g_2(\cdot)$, and $g_3(\cdot)$ in (4).

3 Monotonicity of the optimal cost and an invariant set

In this section, we will show how to choose final weighting functions $g_1(\cdot)$, $g_2(\cdot)$, and $g_3(\cdot)$ in the cost function (4) so that the optimal cost of each horizon is nonincreasing as the horizon slides forward with time. The nonincreasing monotonicity of the optimal cost plays an important role in proving the closed-loop stability of the proposed RHC.

Adding $*$ to the cost function (3) and representing it in terms of initial states and inputs, we denote the optimal cost by $J^*(x_{t_0}, u_{t_0}, t_0, t_1)$. We first show that $J^*(x_{t_0}, u_{t_0}, t_0, t_1)$ is a nonincreasing function with respect to t_1 under some condition on the terminal weighting functions and then consider the monotonicity of the receding horizon cost, i.e., $J^*(x_t, u_t, t, t+T)$.

Theorem 1 *Assume that $g_1(\cdot)$, $g_2(\cdot)$, and $g_3(\cdot)$ in (3) satisfy the following inequality for all x_σ and u_σ*:

$$
\begin{aligned}
&l_1\big(x(\sigma)\big) + l_2\big(k(x_\sigma, u_\sigma)\big) \\
&\quad + \left(\frac{\partial g_1}{\partial x}\right)^T f\big(x(\sigma), x(\sigma-h_x), k(x_\sigma, u_\sigma), u(\sigma-h_u)\big) \\
&\quad + g_2\big(x(\sigma)\big) - g_2\big(x(\sigma-h_x)\big) \\
&\quad + g_3\big(k(x_\sigma, u_\sigma)\big) - g_3\big(u(\sigma-h_u)\big) \le 0,
\end{aligned}
\tag{5}
$$

where $k(\cdot,\cdot)$ is a certain functional of x_σ and u_σ. Then the optimal cost $J^*(x_\tau, u_\tau, \tau, \sigma)$ satisfies the following relation:

$$
\frac{\partial J^*(x_\tau, u_\tau, \tau, \sigma)}{\partial \sigma} \le 0, \quad \tau \le \sigma. \tag{6}
$$

Proof $u^1(t)$ and $u^2(t)$ denotes the optimal controls to minimize $J(x, u, \tau, \sigma+\Delta)$ and $J(x, u, \tau, \sigma)$, respectively. $x^1(t)$ and $x^2(t)$ are the corresponding state trajectories driven by $u^1(t)$ and $u^2(t)$, respectively. It follows with these notations that we have

$$
\begin{aligned}
&\frac{\partial J^*(x_\tau, u_\tau, \tau, \sigma)}{\partial \sigma} \\
&= \lim_{\Delta\to 0} \frac{1}{\Delta}\big\{J^*(x_\tau, u_\tau, \tau, \sigma+\Delta) \\
&\quad - J^*(x_\tau, u_\tau, \tau, \sigma)\big\} \\
&= \lim_{\Delta\to 0} \frac{1}{\Delta}\bigg\{\int_\tau^\sigma \big[l_1\big(x^1(t)\big) + l_2\big(u^1(t)\big)\big]\,dt \\
&\quad + J^*\big(x_\sigma^1, u_\sigma^1, \sigma, \sigma+\Delta\big) \\
&\quad - \int_\tau^\sigma \big[l_1\big(x^2(t)\big) + l_2\big(u^2(t)\big)\big]\,dt - g_1\big(x^2(\sigma)\big) \\
&\quad - \int_{\sigma-h_x}^\sigma g_2\big(x^2(t)\big)\,dt - \int_{\sigma-h_u}^\sigma g_3\big(u^2(t)\big)\,dt\bigg\}.
\end{aligned}
\tag{7}
$$

If $u^1(\cdot)$ is replaced by $u^2(\cdot)$ up to σ and a certain feedback control $u^1(t) = k(x_t, u_t)$ for $t \ge \sigma$, then the following inequality is obtained:

$$
\begin{aligned}
&\frac{\partial J^*(x_\tau, u_\tau, \tau, \sigma)}{\partial \sigma} \\
&\le \lim_{\Delta\to 0} \frac{1}{\Delta}\bigg\{\int_\tau^\sigma \big[l_1\big(x^2(t)\big) + l_2\big(u^2(t)\big)\big]\,dt \\
&\quad + J\big(x^2, k(\cdot,\cdot), \sigma, \sigma+\Delta\big) \\
&\quad - \int_\tau^\sigma \big[l_1\big(x^2(t)\big) + l_2\big(u^2(t)\big)\big]\,dt - g_1\big(x^2(\sigma)\big) \\
&\quad - \int_{\sigma-h_x}^\sigma g_2\big(x^2(t)\big)\,dt - \int_{\sigma-h_u}^\sigma g_3\big(u^2(t)\big)\,dt\bigg\} \\
&= l_1\big(x^2(\sigma)\big) + l_2\big(k\big(x_\sigma^2, u_\sigma^2\big)\big) \\
&\quad + \left(\frac{\partial g_1}{\partial x}\right)^T f\big(x^2(\sigma), x^2(\sigma-h_x), \\
&\qquad\qquad k\big(x_\sigma^2, u_\sigma^2\big), u^2(\sigma-h_u)\big) \\
&\quad + g_2\big(x^2(\sigma)\big) - g_2\big(x^2(\sigma-h_x)\big) + g_3\big(k\big(x_\sigma^2, u_\sigma^2\big)\big) \\
&\quad - g_3\big(u^2(\sigma-h_u)\big) \le 0,
\end{aligned}
$$

where the first inequality comes from the optimality of the cost function. It is noted that, since $u(t) = k(x_t, u_t)$ for $t \ge \sigma$ may not be optimal on $[\sigma,\ \sigma+\Delta]$, the resulting state trajectory on $[\sigma,\ \sigma+\Delta]$ is neither x^1nor x^2. However, we still used x^2 instead of introducing a new variable because Δ is close to zero and hence a real trajectory approaches x^2.
This completes the proof. □

For linear systems, the cost monotonicity conditions have been well known to be useful for showing the stability of the RHC [4, 5, 22, 24]. The inequality (5) for a nonlinear time-delay system is a generalization of the existing cost monotonicity conditions. It is also noted that the inequality (5) should hold without respect to x_σ and u_σ. To the end, $g_1(\cdot)$, $g_2(\cdot)$, $g_3(\cdot)$, and $k(\cdot,\cdot)$ in (5) should be properly chosen, which will be discussed in the next section.

The monotonicity of the receding horizon cost function $J^*(x_t,u_t,t,t+T)$ can be easily derived from the inequality (6). Partitioning the horizon $[t,\ t+T]$ into $[t,\ t+\theta]$ and $[t+\theta,\ t+T]$ and using the inequality (6) yield

$$\begin{aligned} &J^*(x_t,u_t,t,t+T)\\ &\quad\geq \int_t^{t+\theta} \big[l_1\big(x(\tau)\big)+l_2\big(u^*(\tau)\big)\big]\,d\tau\\ &\qquad + J^*(x_{t+\theta},u_{t+\theta},t+\theta,t+T+\theta), \end{aligned} \tag{8}$$

where $u^*(\cdot)$ denotes the optimal control. Dividing both sides of (8) by θ and taking the limit as $\theta\to 0$, we have

$$\frac{dJ^*(x_t,u_t,t,t+T)}{dt}\leq -l_1\big(x(t)\big)-l_2\big(u^*(t)\big), \tag{9}$$

which means that $J^*(x_t,u_t,t,t+T)$ is strictly decreasing except for zero states and inputs.

In addition to the cost monotonicity condition, an invariant set including the origin is necessary for guaranteeing the stability when the constraints are imposed. Once the state enters the invariant set and a simple feedback control $u(t)=k(x_t,u_t)$ is applied, the state stays inside forever while meeting the constraints. If the state is far away from the origin, we have only to steer it to the invariant set. In this paper, we employ the following invariant set:

$$\mathcal{E}_F^t \triangleq \big\{x_t,u_t \,|\, J_F(x_t,u_t)\leq\gamma\big\}, \tag{10}$$

for a given positive constant γ. The invariant property of the set (10) can be easily shown as follows: According to the inequality (5), we have

$$\frac{dJ_F(x_t,u_t)}{dt}\leq -l_1\big(x(t)\big)-l_2\big(k(x_t,u_t)\big), \tag{11}$$

which means that $J_F(\cdot,\cdot)$ decreases with time and thus the trajectories generated from states and inputs in $\mathcal{E}_F^t$ stay inside. It is noted that all states and inputs in $\mathcal{E}_F^t$ should belong to $\mathcal{U}$ and $\mathcal{X}$ in (2) in order for any trajectories in $\mathcal{E}_F^t$ to stay inside while meeting the constraints. However, it may be difficult to do so since constraints $\mathcal{U}$ and $\mathcal{X}$ in (2) take the pointwise form based on the L_∞ norm bound while $\mathcal{E}_F^t$ does the intervalwise form based on the L_2 norm bound. Due to integration terms of the terminal weighting function (4), large states and inputs can happen in a moment even with a small γ and then violate the constraints at some time points. In order to overcome this problem, a subset of $\mathcal{E}_F^t$ is considered, where constraints are satisfied.

To begin with, we introduce an additional set of the inputs and the states,

$$\begin{aligned} \mathcal{E}_C^t \triangleq \big\{&x_t,u_t \,|\, x(\tau)\in\mathcal{X} \text{ for } \tau\in[t-h_x,t],\\ &u(\tau)\in\mathcal{U} \text{ for } \tau\in[t-h_u,t]\big\}, \end{aligned} \tag{12}$$

where all trajectories generated from the inputs and states in (12) satisfy the constraints (2). Using the set (12), we can redefine the terminal invariant set considering constraints (2) as

$$\mathcal{E}^t \triangleq \mathcal{E}_C^t \cap \mathcal{E}_F^t. \tag{13}$$

It implies that any trajectories starting from the inputs and the states in $\mathcal{E}^t$ stay inside $\mathcal{E}_F^t$ while meeting the constraints. In other words, once the state enters $\mathcal{E}^t$ instead of $\mathcal{E}_F^t$, there is a certain control such as a stabilizable feedback control $u(t)=k(x_t,u_t)$, which makes the state stay inside $\mathcal{E}_F^t$ forever while meeting the constraints. How to choose $\mathcal{E}_C^t$ will be discussed later on for a certain type of nonlinear time-delay systems.

4 Stability of the RHC

A cost monotonicity condition and an invariant set introduced in previous sections are put together to achieve the stability of the RHC. To begin with, the following theorem summarizes the stability of the RHC for unconstrained systems, i.e., $\mathcal{U}=\Re^n$, $\mathcal{X}=\Re^m$.

Theorem 2 *If* $\frac{\partial J^*(x_t,u_t,t,\sigma)}{\partial\sigma}|_{\sigma=t+T}\leq 0$, *the unconstrained system* (1) *with the proposed RHC is asymptotically stable.*

Proof In the previous section, we showed that $\frac{\partial J^*(x_t,u_t,t,\sigma)}{\partial\sigma}|_{\sigma=t+T}\leq 0$ implies $\frac{dJ^*(x_t,u_t,t,t+T)}{dt}\leq$

$-l_1(x(t))-l_2(u^*(t))$. Since $J^*(x_t,u_t,t,t+T)$ is positive for nonzero states and inputs and strictly decreasing with time according to (9), it can serve as a Lyapunov function. So, we can conclude that all states go to the origin, and hence the closed-loop system is asymptotically stable. This completes the proof. □

For constrained systems, i.e., $\mathcal{U} \subsetneqq \Re^n$, $\mathcal{X} \subsetneqq \Re^m$, an invariant set in the previous section is utilized to guarantee the feasibility, or obtain a stabilizing RHC for all the time. The optimization problem for the RHC can be formulated as follows:

$$\begin{aligned} &\min_u J(x,u,t,t+T), \\ &\text{subject to} \quad (1), (2), (u_{t+T}, x_{t+T}) \in \mathcal{E}^{t+T}. \end{aligned} \tag{14}$$

Each time, the RHC is computed by solving an optimization problem (14) on the finite future horizon and choosing only the current control. This procedure then repeats at the next time. Collecting the above results, we have the following theorem.

Theorem 3 *Assume that there exist $g_1(\cdot)$, $g_2(\cdot)$, $g_3(\cdot)$, and $k(\cdot,\cdot)$ satisfying the condition (5) for all x_σ and u_σ. If the optimization problem (14) is feasible at the initial time, then the system (1) with the RHC is asymptotically stable.*

Proof The terminal condition $(u_{t+T},\ x_{t+T}) \in \mathcal{E}^{t+T}$ guarantees that the inequality (5) still holds for constrained systems. In other words, the terminal feedback control $u(t+T)=k(x_{t+T},u_{t+T})$ in (5) is feasible due to $(u_{t+T},\ x_{t+T}) \in \mathcal{E}^{t+T}$. The stability can be shown in a similar way to the Theorem 2 for unconstrained systems.

If the optimization problem (14) has a solution at the initial time, this solution is one of stabilizing controls that satisfy constraints. An open loop optimal control for the optimization problem (14) over the horizon $[t,\ t+T]$ and a feedback control $u(t)=k(x_t,u_t)$ after $t+T$ compose a stabilizing control working for $[t,\ \infty]$. In other words, we have at least one stabilizing solution for all the time. By optimizing the cost function each time, we can obtain the better solution. This completes the proof. □

Until now, we have dealt with general nonlinear systems. In the next section, the systems and the cost functions are functions are specified to get implementable solutions.

5 A special class of nonlinear time-delay systems

It is difficult to find $g_1(\cdot)$, $g_2(\cdot)$, $g_3(\cdot)$, and $k(\cdot,\cdot)$ in Theorem 1 for general nonlinear time-delay systems (1). In this section, a special class of nonlinear time-delay systems is introduced, for which $g_1(\cdot)$, $g_2(\cdot)$, $g_3(\cdot)$, and $k(\cdot,\cdot)$ satisfying the inequality condition (5) can be easily obtained by solving a linear matrix inequality(LMI) problem.

Consider the following nonlinear time-delay systems

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Df_1\big(x(t)\big) + A_1x(t-h_x) \\ &\quad + D_1f_2\big(x(t-h_x)\big) + Bu(t) \\ &\quad + Ef_3\big(u(t)\big) + B_1u(t-h_u) \\ &\quad + E_1f_4\big(u(t-h_u)\big), \end{aligned}$$

where $x(t)=\phi(t)$ for $t\in[-h_x,0\,]$, $u(t)=\psi(t)$ for $t\in[-h_u,0\,]$, and $f_i(\cdot)$ satisfy

$$\begin{aligned} &\|f_i(x)-L_ix\|_2 \le \|M_ix\|_2, \\ &f_i(0)=0, \end{aligned} \tag{15}$$

for $i=1,2,3,4$. Assume that $l_1(\cdot)$ and $l_2(\cdot)$ in (3) have quadratic forms: $l_1(x(t))=x^T(t)Qx(t)$, $l_2(u(t))=u^T(t)Ru(t)$, where $Q=Q^T>0$ and $R=R^T>0$.

Theorem 4 *If there exist $X>0$, Y_1, Y_2, Y_3, Z, S, $\epsilon_1>0$, $\epsilon_2>0$, $\epsilon_3>0$, and $\epsilon_4>0$ such that*

$$\begin{bmatrix} P_{11} & P_{1,2} & P_{1,3} & X & X & Y_1^T & Y_1^T & XM_1^T & O & Y_1^TM_3^T & O \\ \star & -Z & O & O & O & Y_2^T & Y_2^T & O & ZM_2^T & Y_2^TM_3^T & O \\ \star & \star & -S & O & O & Y_3^T & Y_3^T & O & O & Y_3^TM_3^T & SM_4^T \\ \star & \star & \star & & & & & \Sigma & & & \end{bmatrix} < O \tag{16}$$

where Σ, P_{11}, P_{12}, *and* P_{13} *are given by*

$$\Sigma = \mathrm{diag}\big(-Z, -Q^{-1}, -S, -R^{-1}, \\ -\epsilon_1 I, -\epsilon_2 I, -\epsilon_3 I, -\epsilon_4 I\big)$$

$$P_{11} = \big((A + DL_1)X + (B + EL_3)Y_1\big)^T \\ + \big((A + DL_1)X + (B + EL_3)Y_1\big) \\ + \epsilon_1 DD^T + \epsilon_2 D_1 {D_1}^T + \epsilon_3 EE^T + \epsilon_4 E_1 E_1^T,$$

$$P_{12} = (A_1 + D_1 L_2)Z + (B + EL_3)Y_2,$$

$$P_{13} = (B_1 + E_1 L_4)S + (B + EL_3)Y_3,$$

then the inequality condition (1) *is satisfied with* $g_1(x(t)) = x^T(t)Q_f x(t)$, $g_2(x(t)) = x^T(t)Q_h x(t)$, $g_3(x(t)) = u^T(t)R_h u(t)$, *and* $k(x_t, u_t) = K_1 x(t) + K_2 x(t - h_x) + K_3 u(t - h_u)$. *Furthermore,* Q_f, Q_h, R_h, K_1, K_2, *and* K_3 *are obtained from*

$$Q_f = X^{-1}, \quad Q_h = Z^{-1}, \quad R_h = S^{-1}, \\ K_1 = Y_1 X^{-1}, \quad K_2 = Y_2 Z^{-1}, \quad K_3 = Y_3 S^{-1}.$$

Proof By utilizing the well-known inequality $x^T y + y^T x \le \epsilon x^T x + \epsilon^{-1} y^T y$ for any positive ϵ, we can easily obtain the following inequality:

$$l_1\big(x(\sigma)\big) + l_2\big(k(x_\sigma, u_\sigma)\big) + \left(\frac{\partial g_1}{\partial x}\right)^T f\big(x(\sigma), x(\sigma - h_x), k(x_\sigma, u_\sigma), u(x - h_u)\big) \\ + g_2\big(x(\sigma)\big) - g_2\big(x(\sigma - h_x)\big) + g_3\big(k(x_\sigma, u_\sigma)\big) - g_3\big(u(\sigma - h_u)\big) \\ \le \begin{bmatrix} x(\sigma) \\ x(\sigma - h_x) \\ u(\sigma - h_u) \end{bmatrix}^T \begin{bmatrix} (1,1) & (1,2) & (1,3) \\ \star & (2,2) & (2,3) \\ \star & \star & (3,3) \end{bmatrix} \begin{bmatrix} x(\sigma) \\ x(\sigma - h_x) \\ u(\sigma - h_u) \end{bmatrix}, \tag{17}$$

where block matrices indexed by a pair of numbers are given by

$$(1,1) \triangleq \big(A + DL_1 + (B + EL_3)K_1\big)^T Q_f \\ + Q_f\big(A + DL_1 + (B + EL_3)K_1\big) \\ + \epsilon_1 Q_f DD^T Q_f + \epsilon_2 Q_f D_1 {D_1}^T Q_f \\ + \epsilon_3 Q_f EE^T Q_f + \epsilon_4 Q_f E_1 E_1^T Q_f \\ + \epsilon_1^{-1} M_1^T M_1 + K_1^T\big(R + R_h \\ + \epsilon_3^{-1} M_3^T M_3\big)K_1 + Q_h + Q$$

$$(1,2) \triangleq Q_f\big(A_1 + D_1 L_2 + (B + EL_3)K_2\big) \\ + K_1^T\big(R + R_h + \epsilon_3^{-1} M_3^T M_3\big)K_2$$

$$(1,3) \triangleq Q_f\big(B_1 + E_1 L_4 + (B + EL_3)K_3\big) \\ + K_1^T\big(R + R_h + \epsilon_3^{-1} M_3^T M_3\big)K_3$$

$$(2,2) \triangleq -Q_h + \epsilon_2^{-1} M_2^T M_2 \\ + K_2^T\big(R + R_h + \epsilon_3^{-1} M_3^T M_3\big)K_2$$

$$(2,3) \triangleq K_2^T\big(R + R_h + \epsilon_3^{-1} M_3^T M_3\big)K_3$$

$$(3,3) \triangleq -R_h + \epsilon_4^{-1} M_4^T M_4 \\ + K_3^T\big(R + R_h + \epsilon_3^{-1} M_3^T M_3\big)K_3.$$

If the 3-by-3 block matrix in (17) is negative definite, we can say that the cost monotonicity condition is satisfied. From the Schur complement method, the negative definiteness of the 3-by-3 block matrix in (17) is equivalent to

$$\begin{bmatrix} P_{11} & P_{12} & P_{13} & I & I & K_1^T & K_1^T & M_1^T & O & K_1^T M_3^T & O \\ \star & -Q_h & O & O & O & K_2^T & K_2^T & O & M_2^T & K_2^T M_3^T & O \\ \star & \star & -R_h & O & O & K_3^T & K_3^T & O & O & K_3^T M_3^T & M_4^T \\ \star & \star & \star & & & & \Gamma & & & & \end{bmatrix} < O \tag{18}$$

where Γ, P_{11}, P_{12}, and P_{13} are defined by

$$\Gamma \triangleq \mathrm{diag}\big(-Q_h^{-1}, -Q^{-1}, -R_h^{-1}, -R^{-1}, -\epsilon_1 I, -\epsilon_2 I, -\epsilon_3 I, -\epsilon_4 I\big)$$

$$P_{11} \triangleq \big(A + DL_1 + (B + EL_3)K\big)^T Q_f + Q_f\big(A + DL_1 + (B + EL_3)K\big) + \epsilon_1 Q_f DD^T Q_f + \epsilon_2 Q_f D_1 {D_1}^T Q_f + \epsilon_3 Q_f EE^T Q_f + \epsilon_4 Q_f E_1 E_1^T Q_f,$$

$$P_{12} \triangleq Q_f\big(A_1 + D_1 L_2 + (B + EL_3)K_2\big),$$

$$P_{13} \triangleq Q_f\big(B_1 + E_1 L_4 + (B + EL_3)K_3\big).$$

Pre- and post-multiplying the inequality (18) by $\mathrm{diag}(Q_f^{-1}, Q_h^{-1}, R_h^{-1}, I, I, I, I, I, I, I, I)$ and performing the change of variables such that $X \triangleq Q_f^{-1}$, $Z \triangleq Q_h^{-1}$, $S \triangleq R_h^{-1}$, $Y_1 \triangleq K_1 Q_f^{-1}$, $Y_2 \triangleq K_2 Q_h^{-1}$, and $Y_3 \triangleq K_3 R_h^{-1}$, the inequality (18) is equivalently changed into the LMI (16). This completes the proof. □

Remark 1 For linear systems with state or input delay case, *i.e.*, $D = E = D_1 = E_1 = O$, the LMI (16) is exactly the same with the LMI in [24] and [22].

From now on, we discuss how to choose $\mathcal{E}_C^t$ that is introduced in the previous section. For simple computation, $\mathcal{E}_C^t$ is taken as follows:

$$\mathcal{E}_C^t = \big\{x_t, u_t | x^T(\tau) Q_f x(\tau) \le \gamma_x, \tau \in [t - h_x, t], u^T(\tau) R_h u(\tau) \le \gamma_u, \tau \in [t - h_u, t]\big\}. \tag{19}$$

At least, γ_x and γ_u should be chosen such that all states and inputs in $\mathcal{E}_C^t$ belong to $\mathcal{X}$ and $\mathcal{U}$, respectively. Such γ_x and γ_u are denoted by γ_{x1} and γ_{u1}. What remains to check is whether the trajectories stay inside.

If the gain K_3 satisfies the following inequality:

$$\begin{bmatrix} \frac{1}{3} R_h & K_3^T \\ K_3 & R_h^{-1} \end{bmatrix} > O, \tag{20}$$

we can choose γ_x and γ_u such that the following inequality holds:

$$\begin{aligned} u^T(t) R_h u(t) &= \big(K_1 x(t) + K_2 x(t-h) + K_3 u(t-h)\big)^T R_h \big(K_1 x(t) + K_2 x(t-h) + K_3 u(t-h)\big) \\ &\le 3\lambda_{\max}\big(Q_f^{-\frac{1}{2}} K_1^T R_h K_1 Q_f^{-\frac{1}{2}}\big)\gamma_x + 3\lambda_{\max}\big(Q_f^{-\frac{1}{2}} K_2^T R_h K_2 Q_f^{-\frac{1}{2}}\big)\gamma_x + 3\lambda_{\max}\big(R_h^{-\frac{1}{2}} K_3^T R_h K_3 R_h^{-\frac{1}{2}}\big)\gamma_u \\ &\le \gamma_u \end{aligned} \tag{21}$$

for all $(x_t, u_t) \in \mathcal{E}_C^t$. These γ_x and γ_u are denoted by γ_{x2} and γ_{u2}. The inequality (21) tells us that $\mathcal{E}_C^t$ is invariant with respect to the inputs, and thus input trajectories satisfy the constraints $\mathcal{U}$ if $\gamma_{u2} \le \gamma_{u1}$. If we take the final γ_x and γ_u as $\min\{\gamma_{x1}, \gamma_{x2}\}$ and $\min\{\gamma_{u1}, \gamma_{u2}\}$, respectively, and set γ in $\mathcal{E}_F^t$ to γ_x, we can construct a set $\mathcal{E}^t$. The reason for $\gamma_x = \gamma$ is that $\mathcal{E}_C^t$ is invariant with respect to the states, and thus all state trajectories satisfy the constraints $\mathcal{X}$. As mentioned in the Sect. 3, all trajectories starting from the inputs and the states in the set $\mathcal{E}^t$ stay inside $\mathcal{E}_F^t$ while meeting the constraints. The condition on K_3 (20) can be considered in computing the LMI (16).

6 Numerical example

In this section, a numerical example is presented to illustrate the performance of the proposed RHC. Consider a nonlinear system given by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0.2 & 0.1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + 0.6 \begin{bmatrix} x_1(t - h_x)\cos^2(0.5 x_1(t - h_x)) \\ x_2(t - h_x) \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} u(t) + 0.5 \begin{bmatrix} 0 \\ u(t - h_u)\cos^2(0.5 u(t - h_u)) \end{bmatrix}.$$

We can see that this system belongs to the class considered in Sect. 5. It is noted that the corresponding model parameters are given by

$$
\begin{aligned}
&f_1 = f_3 = O, \qquad A_1 = B_1 = O, \qquad D_1 = 0.6,\\
&E_1 = 0.5, \qquad M_2 = \begin{bmatrix}\frac{1}{2} & 0\end{bmatrix}, \qquad M_4 = \frac{1}{2},\\
&f_2(x) = \begin{bmatrix} x_1 \cos^2(x_1) \\ x_2 \end{bmatrix},\\
&f_4(u) = \begin{bmatrix} 0 \\ u\cos^2(u) \end{bmatrix}, \qquad L_2 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix},\\
&L_4 = \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix},
\end{aligned} \tag{22}
$$

and the initial values given by

$$x(t) = \begin{bmatrix} -0.1t + 0.5 \\ -5t + 0.6 \end{bmatrix}, \qquad -h_x \le t \le 0,$$

$$u(t) = -0.8t + 1.6, \qquad -h_u \le t \le 0.$$

The delay size h_x and h_u are set to 0.5 and 0.4, respectively. The horizon length T is chosen to be 0.8. It is noted that this system is unstable with zero input. For $Q = I_{2\times 2}$ and $R = 1$, the final weighting matrices and the corresponding feedback gains guaranteeing the cost monotonicity are obtained by solving the LMI (16) in Theorem 4, which is computed as

$$Q_f = \begin{bmatrix} 17.5913 & 8.1286 \\ 8.1286 & 4.8030 \end{bmatrix},$$

$$Q_h = \begin{bmatrix} 1.8576 & 0.3229 \\ 0.3229 & 0.5967 \end{bmatrix}, \qquad R_h = 0.6850,$$

$$K_1 = -\begin{bmatrix} 5.9016 & 3.4254 \end{bmatrix},$$

$$K_2 = -\begin{bmatrix} -0.0036 & 0.2114 \end{bmatrix}, \qquad K_3 = -0.0736.$$

The input and state constraints are given as follows:

$$\mathcal{U} = \{u \mid -4.5 \le u(t) \le 4.5\},$$

$$\mathcal{X} = \{x \mid -0.53 \le x_1(t) \le 0.53, \ -1.2 \le x_2(t) \le 1.2\}.$$

From this constraints, $\gamma_x = 1.0771$ and $\gamma_u = 5.6007$ are obtained according to the results of the Sect. 5.

All simulation was fulfilled in MATLAB. The step size for numerical integration, i.e., the fourth-order Runge–Kutta method, is chosen to be 0.01. For receding horizon implementation, the state measurement is taken at the sample time of 0.04 second. For computation, it takes about 11.3 ms to obtain the RHC at each sampling points by using a computer with a 2.26 GHz quad-core processor. If the target system is not so fast, we believe that the proposed control works well. This computation burden can be alleviated by the recent fast computer technology. For constrained optimization, the command "fmincon" in the optimization toolbox was used. All MATLAB source codes for this simulation are available at the website [25].

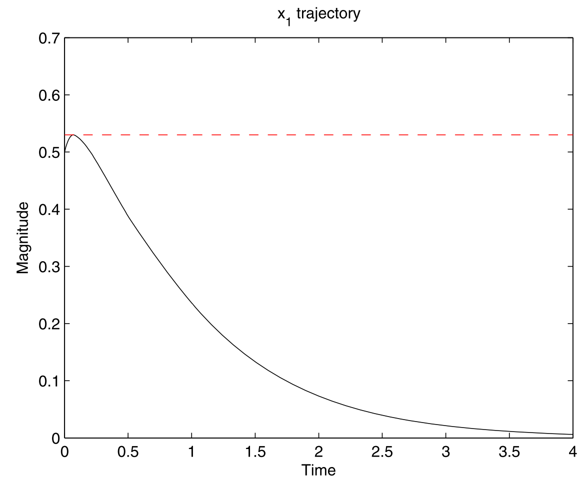

Fig. 1 The trajectory of the first component of the state

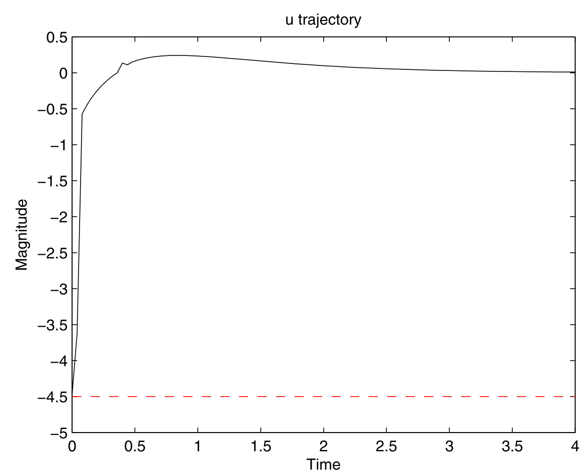

Fig. 2 The trajectory of the input

From Figs. 1 and 2, we can see the proposed constrained RHC stabilizes a nonlinear time-delay system with satisfying the state and input constraints. Figure 3 shows the trajectory of the optimal receding horizon cost J^*.

It is shown that the optimal receding horizon cost is monotonically decreasing against time and converges to zero. This monotonicity implies that the proposed

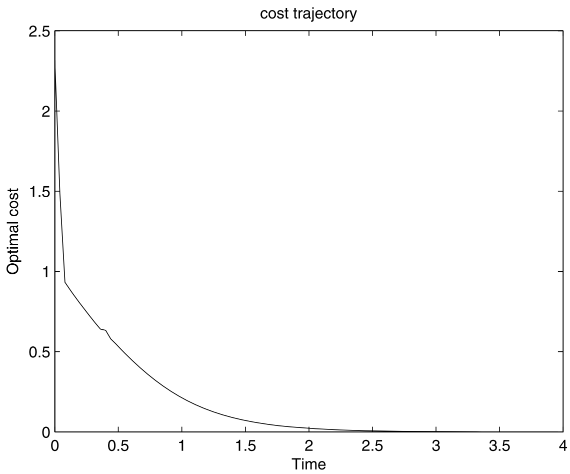

Fig. 3 The trajectory of optimal cost

constrained RHC provides a stabilizing control for a nonlinear time-delay system with state and input constraints.

7 Conclusions

This paper has first presented a stabilizing RHC for nonlinear time-delay systems with input and state constraints. An inequality condition on the terminal weighting functions was presented, under which the optimal cost has the nonincreasing monotonicity and offers an invariant set. Putting together the cost monotonicity condition and the invariant set, we have shown the stability of the RHC. A special class of nonlinear time delay systems and cost functions were introduced to show the feasibility of the proposed conditions and then illustrate the performance of the proposed RHC. It is shown in a numerical example that the proposed RHC guarantees the closed-loop stability and satisfies the input and state constraints for nonlinear systems with both input and state delays.

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